

Supersymmetry algebra cohomology III: Primitive elements in four and five dimensions

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Abstract

The primitive elements of the supersymmetry algebra cohomology as defined in a previous paper are computed for standard supersymmetry algebras in four and five dimensions, for all signatures of the metric and any number of supersymmetries.

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1 Introduction

This paper relates to supersymmetry algebra cohomology as defined in [1], for supersymmetry algebras in $D = 4$ and $D = 5$ dimensions of translational generators

P_a ($a = 1, \dots, D$) and supersymmetry generators $Q_{\underline{\alpha}}^i$ ($i = 1, \dots, N$, $\underline{\alpha} = \underline{1}, \dots, \underline{4}$) of the form

$$[P_a, P_b] = 0, \quad [P_a, Q_{\underline{\alpha}}^i] = 0, \quad \{Q_{\underline{\alpha}}^i, Q_{\underline{\beta}}^j\} = M^{ij} (\Gamma^a C^{-1})_{\underline{\alpha}\underline{\beta}} P_a \quad (1.1)$$

where M^{ij} are the entries of an $N \times N$ matrix M given by

$$D = 4 : M = \begin{pmatrix} -i & 0 & \cdots & 0 \\ 0 & -i & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -i \end{pmatrix}, \quad D = 5 : M = \begin{pmatrix} \sigma_2 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_2 \end{pmatrix} \quad (1.2)$$

with i denoting the imaginary unit and σ_2 denoting the second Pauli-matrix (hence, in the $D = 5$ we consider $N = 2, 4, 6, \dots$).

The object of this paper is the determination of the primitive elements of the supersymmetry algebra cohomology for these supersymmetry algebras (1.1) for all signatures $(t, D - t)$ ($t = 0, \dots, D$) of the Clifford algebra of the gamma matrices Γ^a . According to the definition given in [1], these primitive elements are the representatives of the cohomology $H_{\text{gh}}(s_{\text{gh}})$ of the coboundary operator

$$s_{\text{gh}} = -\frac{1}{2} M^{ij} (\Gamma^a C^{-1})_{\underline{\alpha}\underline{\beta}} \xi_{\underline{\alpha}}^i \xi_{\underline{\beta}}^j \frac{\partial}{\partial c^a} \quad (1.3)$$

in the space Ω_{gh} of polynomials in translation ghosts c^a and supersymmetry ghosts $\xi_{\underline{\alpha}}^i$ corresponding to the translational generators P_a and the supersymmetry generators $Q_{\underline{\alpha}}^i$, respectively,

$$\Omega_{\text{gh}} = \left\{ \sum_{p=0}^D \sum_{n=0}^r c^{a_1} \dots c^{a_p} \xi_{\underline{i}_1}^{\underline{\alpha}_1} \dots \xi_{\underline{i}_n}^{\underline{\alpha}_n} a_{\underline{\alpha}_1 \dots \underline{\alpha}_n a_1 \dots a_p}^{i_1 \dots i_n} \mid a_{\underline{\alpha}_1 \dots \underline{\alpha}_n a_1 \dots a_p}^{i_1 \dots i_n} \in \mathbb{C}, \quad r = 0, 1, 2, \dots \right\}. \quad (1.4)$$

For signatures $(1, 3)$, $(2, 2)$ and $(3, 1)$ in $D = 4$ and signatures $(2, 3)$ and $(3, 2)$ in $D = 5$ the supersymmetry ghosts are Majorana spinors, for signatures $(0, 4)$ and $(4, 0)$ in $D = 4$ and signatures $(0, 5)$, $(1, 4)$, $(4, 1)$ and $(5, 0)$ in $D = 5$ they are symplectic Majorana spinors, cf. sections 2 and 4 of [1]. We note that for signature $(2, 2)$ in $D = 4$ each Majorana spinor consists of two Majorana-Weyl spinors of opposite chirality and for signatures $(0, 4)$ and $(4, 0)$ each symplectic Majorana spinor consists of two symplectic Majorana-Weyl spinors of opposite chirality. N denotes in all cases the number of Majorana or symplectic Majorana supersymmetry ghosts (hence, for signature $(2, 2)$ and $N = 1$ one has one Majorana supersymmetry ghost and thus two Majorana-Weyl supersymmetry ghosts etc.).

Analogously to the strategy applied in [2] in two and three dimensions, we shall first compute $H_{\text{gh}}(s_{\text{gh}})$ in $D = 4$ explicitly in a particular spinor representation and then covariantize the results to make them independent of the spinor representation.

Afterwards the results in $D = 5$ are derived by means of the results in $D = 4$. The particular spinor representations are defined by

$$\begin{aligned}\Gamma_1 &= k_1 \sigma_1 \otimes \sigma_0, \quad \Gamma_2 = k_2 \sigma_2 \otimes \sigma_0, \quad \Gamma_3 = k_3 \sigma_3 \otimes \sigma_1, \quad \Gamma_4 = k_4 \sigma_3 \otimes \sigma_2, \\ D = 4 : \quad \hat{\Gamma} &= \sigma_3 \otimes \sigma_3, \quad C = \sigma_2 \otimes \sigma_1, \\ D = 5 : \quad \Gamma_5 &= k_5 \sigma_3 \otimes \sigma_3, \quad C = \sigma_1 \otimes \sigma_2\end{aligned}\tag{1.5}$$

with

$$k_a = \begin{cases} i & \text{for } a \leq t \\ 1 & \text{for } a > t \end{cases} .\tag{1.6}$$

As in [2] we shall use the notation \sim for equivalence in $H_{\text{gh}}(s_{\text{gh}})$, i.e. for $\omega_1, \omega_2 \in \Omega_{\text{gh}}$ the notation $\omega_1 \sim \omega_2$ means $\omega_1 - \omega_2 = s_{\text{gh}}\omega_3$ for some $\omega_3 \in \Omega_{\text{gh}}$:

$$\omega_1 \sim \omega_2 \quad :\Leftrightarrow \quad \exists \omega_3 : \omega_1 - \omega_2 = s_{\text{gh}}\omega_3 \quad (\omega_1, \omega_2, \omega_3 \in \Omega_{\text{gh}}).\tag{1.7}$$

Furthermore the paper uses terminology, notation and conventions introduced in [1].

2 Primitive elements in four dimensions

2.1 $H_{\text{gh}}(s_{\text{gh}})$ in particular spinor representations

We shall first compute $H_{\text{gh}}(s_{\text{gh}})$ for $D = 4$ in the particular spinor representations (1.5). In order to do this for all signatures $(t, 4-t)$ at once we introduce the following translation ghost variables:

$$\begin{aligned}\tilde{c}^1 &= -\frac{1}{2}(k_1 c^1 + i k_2 c^2), \quad \tilde{c}^2 = -\frac{1}{2}(k_1 c^1 - i k_2 c^2), \\ \tilde{c}^3 &= -\frac{1}{2}(k_3 c^3 + i k_4 c^4), \quad \tilde{c}^4 = \frac{1}{2}(k_3 c^3 - i k_4 c^4).\end{aligned}\tag{2.1}$$

Furthermore, in order to simplify the notation, we denote the components of $\xi_i = (\xi_i^1, \xi_i^2, \xi_i^3, \xi_i^4)$ by $\psi_i, \bar{\psi}^i, -\bar{\chi}^i$ and χ_i :

$$\xi_i = (\xi_i^1, \xi_i^2, \xi_i^3, \xi_i^4) = (\psi_i, \bar{\psi}^i, -\bar{\chi}^i, \chi_i).\tag{2.2}$$

ψ_i and χ_i are the components of Weyl spinors ξ_i^+ with positive chirality, $\bar{\psi}^i$ and $\bar{\chi}^i$ the components of Weyl spinors ξ_i^- with negative chirality ($\xi_i^\pm \hat{\Gamma} = \pm \xi_i^\pm$),

$$\xi_i^+ = (\psi_i, 0, 0, \chi_i), \quad \xi_i^- = (0, \bar{\psi}^i, -\bar{\chi}^i, 0).\tag{2.3}$$

In terms of the ghost variables (2.1) and (2.2), the s_{gh} -transformations of the translation ghost variables are, for all signatures $(t, 4-t)$:

$$s_{\text{gh}}\tilde{c}^1 = \psi_i \bar{\psi}^i, \quad s_{\text{gh}}\tilde{c}^2 = \chi_i \bar{\chi}^i, \quad s_{\text{gh}}\tilde{c}^3 = \psi_i \bar{\chi}^i, \quad s_{\text{gh}}\tilde{c}^4 = \chi_i \bar{\psi}^i.\tag{2.4}$$

2.1.1 $H_{\text{gh}}(s_{\text{gh}})$ for $N = 1$

In order to determine $H_{\text{gh}}(s_{\text{gh}})$ in the case $N = 1$, we first compute the cohomology of s_{gh} in the space Ω^- of polynomials in the ghost variables $\tilde{c}^1, \tilde{c}^2, \tilde{c}^3, \tilde{c}^4, \bar{\psi}^1, \bar{\chi}^1$,

$$\Omega^- = \left\{ \omega \in \Omega_{\text{gh}} \left| \frac{\partial \omega}{\partial \psi_1} = 0 \wedge \frac{\partial \omega}{\partial \chi_1} = 0 \right. \right\}. \quad (2.5)$$

Lemma 2.1.

(i) A polynomial $\omega \in \Omega^-$ is s_{gh} -closed if and only if it is a polynomial in $\bar{\psi}^1, \bar{\chi}^1, \tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1$ and $\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1$:

$$\begin{aligned} \omega \in \Omega^- : s_{\text{gh}} \omega = 0 \Leftrightarrow \omega = & p_1(\bar{\psi}^1, \bar{\chi}^1) \\ & + (\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) p_2(\bar{\psi}^1, \bar{\chi}^1) \\ & + (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) p_3(\bar{\psi}^1, \bar{\chi}^1) \\ & + (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1)(\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) p_4(\bar{\psi}^1, \bar{\chi}^1) \end{aligned} \quad (2.6)$$

with polynomials $p_1(\bar{\psi}^1, \bar{\chi}^1), \dots, p_4(\bar{\psi}^1, \bar{\chi}^1)$ in $\bar{\psi}^1$ and $\bar{\chi}^1$.

(ii) No nonvanishing polynomial in $\bar{\psi}^1, \bar{\chi}^1, \tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1$ and $\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1$ is a coboundary in $H_{\text{gh}}(s_{\text{gh}})$,

$$\begin{aligned} & p_1(\bar{\psi}^1, \bar{\chi}^1) + (\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) p_2(\bar{\psi}^1, \bar{\chi}^1) + (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) p_3(\bar{\psi}^1, \bar{\chi}^1) \\ & + (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1)(\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) p_4(\bar{\psi}^1, \bar{\chi}^1) \sim 0 \\ \Leftrightarrow \forall i \in \{1, 2, 3, 4\} : p_i(\bar{\psi}^1, \bar{\chi}^1) = 0. \end{aligned} \quad (2.7)$$

Proof: We decompose an s_{gh} -cocycle $\omega \in \Omega^-$ into parts ω^p with definite c -degree p . Since s_{gh} decrements the c -degree by one unit, all parts ω^p are s_{gh} -cocycles,

$$s_{\text{gh}} \omega = 0, \quad \omega = \sum_{p=0}^4 \omega^p, \quad N_c \omega^p = p \omega^p \quad \Rightarrow \quad s_{\text{gh}} \omega^p = 0 \quad \forall p \quad (2.8)$$

where N_c denotes the counting operator for the translation ghosts,

$$N_c = c^a \frac{\partial}{\partial c^a}. \quad (2.9)$$

Hence, we can determine the s_{gh} -cocycles in Ω^- separately for the various c -degrees. To determine these s_{gh} -cocycles, we use that s_{gh} acts in terms of the ghost variables $\tilde{c}^1, \tilde{c}^2, \tilde{c}^3, \tilde{c}^4, \psi_1, \bar{\psi}^1, \chi_1, \bar{\chi}^1$ according to

$$s_{\text{gh}} = \psi_1 \bar{\psi}^1 \frac{\partial}{\partial \tilde{c}^1} + \chi_1 \bar{\chi}^1 \frac{\partial}{\partial \tilde{c}^2} + \psi_1 \bar{\chi}^1 \frac{\partial}{\partial \tilde{c}^3} + \chi_1 \bar{\psi}^1 \frac{\partial}{\partial \tilde{c}^4} = \psi_1 D_1 + \chi_1 D_2 \quad (2.10)$$

with

$$D_1 = \bar{\psi}^1 \frac{\partial}{\partial \tilde{c}^1} + \bar{\chi}^1 \frac{\partial}{\partial \tilde{c}^3}, \quad D_2 = \bar{\chi}^1 \frac{\partial}{\partial \tilde{c}^2} + \bar{\psi}^1 \frac{\partial}{\partial \tilde{c}^4}. \quad (2.11)$$

As an element ω of Ω^- neither depends on ψ_1 nor on χ_1 and as D_1 and D_2 do not involve ψ_1 or χ_1 , the cocycle condition $s_{\text{gh}}\omega = (\psi_1 D_1 + \chi_1 D_2)\omega = 0$ imposes $D_1\omega = 0$ and $D_2\omega = 0$. Accordingly, any s_{gh} -cocycle in Ω^- with c -degree p is annihilated both by D_1 and D_2 :

$$s_{\text{gh}}\omega^p = 0, \quad \omega^p \in \Omega^- \quad \Leftrightarrow \quad D_1\omega^p = 0 \quad \wedge \quad D_2\omega^p = 0. \quad (2.12)$$

Part (i) of the lemma is obtained easily by solving $D_1\omega^p = D_2\omega^p = 0$ directly for the various values of p . We shall explicitly spell out the computation only for the most involved case $p = 2$. In this case we have

$$\omega^2 = \tilde{c}^1 \tilde{c}^2 f_{12} + \tilde{c}^1 \tilde{c}^3 f_{13} + \tilde{c}^1 \tilde{c}^4 f_{14} + \tilde{c}^2 \tilde{c}^3 f_{23} + \tilde{c}^2 \tilde{c}^4 f_{24} + \tilde{c}^3 \tilde{c}^4 f_{34} \quad (2.13)$$

with $f_{ij} = f_{ij}(\bar{\chi}^1, \bar{\psi}^1)$. Applying D_1 to ω^2 yields

$$D_1\omega^2 = \tilde{c}^2(\bar{\psi}^1 f_{12} - \bar{\chi}^1 f_{23}) + (\tilde{c}^3 \bar{\psi}^1 - \tilde{c}^1 \bar{\chi}^1) f_{13} + \tilde{c}^4(\bar{\psi}^1 f_{14} + \bar{\chi}^1 f_{34}). \quad (2.14)$$

$D_1\omega^2 = 0$ imposes thus

$$\bar{\psi}^1 f_{12} = \bar{\chi}^1 f_{23}, \quad f_{13} = 0, \quad \bar{\psi}^1 f_{14} = -\bar{\chi}^1 f_{34}$$

which implies

$$f_{12} = \bar{\chi}^1 g_1, \quad f_{23} = \bar{\psi}^1 g_1, \quad f_{13} = 0, \quad f_{14} = \bar{\chi}^1 g_2, \quad f_{34} = -\bar{\psi}^1 g_2 \quad (2.15)$$

for some $g_i = g_i(\bar{\chi}^1, \bar{\psi}^1)$. Using (2.15) in (2.13), we obtain the intermediate result

$$\omega^2 = (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1)(\tilde{c}^2 g_1 + \tilde{c}^4 g_2) + \tilde{c}^2 \tilde{c}^4 f_{24}. \quad (2.16)$$

Applying now D_2 to (2.16) yields

$$D_2\omega^2 = -(\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1)(\bar{\chi}^1 g_1 + \bar{\psi}^1 g_2) + (\tilde{c}^4 \bar{\chi}^1 - \tilde{c}^2 \bar{\psi}^1) f_{24}. \quad (2.17)$$

$D_2\omega^2 = 0$ thus imposes

$$\bar{\chi}^1 g_1 = -\bar{\psi}^1 g_2, \quad f_{24} = 0$$

which implies

$$g_1 = \bar{\psi}^1 p_4, \quad g_2 = -\bar{\chi}^1 p_4, \quad f_{24} = 0 \quad (2.18)$$

for some $p_4 = p_4(\bar{\psi}^1, \bar{\chi}^1)$. Using (2.18) in (2.16), we conclude

$$\omega^2 = (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1)(\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) p_4(\bar{\psi}^1, \bar{\chi}^1) \quad (2.19)$$

which provides the last contribution to ω in (2.6).

Analogously one derives for $p = 0, 3, 4, 1$, respectively:

$$\begin{aligned} \omega^0 &= p_1(\bar{\psi}^1, \bar{\chi}^1), \quad \omega^3 = 0, \quad \omega^4 = 0, \\ \omega^1 &= (\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) p_2(\bar{\psi}^1, \bar{\chi}^1) + (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) p_3(\bar{\psi}^1, \bar{\chi}^1). \end{aligned} \quad (2.20)$$

(2.19) and (2.20) provide part (i) of lemma 2.1. Part (ii) of the lemma holds because cocycles of s_{gh} which neither depend on ψ_1 nor on χ_1 cannot be exact in $H_{\text{gh}}(s_{\text{gh}})$ since every term in (2.10) is linear in ψ_1 or χ_1 (coboundaries contain only terms that depend at least linearly on ψ_1 or χ_1). This completes the proof of lemma 2.1. \blacksquare

A result analogous to lemma 2.1 holds for the cohomology of s_{gh} in the space Ω^+ of polynomials in the ghost variables $\tilde{c}^1, \tilde{c}^2, \tilde{c}^3, \tilde{c}^4, \psi_1, \chi_1$,

$$\Omega^+ = \left\{ \omega \in \Omega_{\text{gh}} \left| \frac{\partial \omega}{\partial \bar{\psi}^1} = 0 \wedge \frac{\partial \omega}{\partial \bar{\chi}^1} = 0 \right. \right\}. \quad (2.21)$$

Lemma 2.2.

(i) A polynomial $\omega \in \Omega^+$ is s_{gh} -closed if and only if it is a polynomial in $\psi_1, \chi_1, \tilde{c}^2\psi_1 - \tilde{c}^3\chi_1$ and $\tilde{c}^1\chi_1 - \tilde{c}^4\psi_1$:

$$\begin{aligned} \omega \in \Omega^+ : s_{\text{gh}}\omega = 0 \Leftrightarrow \omega = & q_1(\psi_1, \chi_1) \\ & + (\tilde{c}^2\psi_1 - \tilde{c}^3\chi_1)q_2(\psi_1, \chi_1) \\ & + (\tilde{c}^1\chi_1 - \tilde{c}^4\psi_1)q_3(\psi_1, \chi_1) \\ & + (\tilde{c}^1\chi_1 - \tilde{c}^4\psi_1)(\tilde{c}^2\psi_1 - \tilde{c}^3\chi_1)q_4(\psi_1, \chi_1) \end{aligned} \quad (2.22)$$

with polynomials $q_1(\psi_1, \chi_1), \dots, q_4(\psi_1, \chi_1)$ in ψ_1 and χ_1 .

(ii) No nonvanishing polynomial in $\psi_1, \chi_1, \tilde{c}^2\psi_1 - \tilde{c}^3\chi_1$ and $\tilde{c}^1\chi_1 - \tilde{c}^4\psi_1$ is a coboundary in $H_{\text{gh}}(s_{\text{gh}})$,

$$\begin{aligned} & q_1(\psi_1, \chi_1) + (\tilde{c}^2\psi_1 - \tilde{c}^3\chi_1)q_2(\psi_1, \chi_1) + (\tilde{c}^1\chi_1 - \tilde{c}^4\psi_1)q_3(\psi_1, \chi_1) \\ & + (\tilde{c}^1\chi_1 - \tilde{c}^4\psi_1)(\tilde{c}^2\psi_1 - \tilde{c}^3\chi_1)q_4(\psi_1, \chi_1) \sim 0 \\ \Leftrightarrow \forall i \in \{1, 2, 3, 4\} : & q_i(\psi_1, \chi_1) = 0. \end{aligned} \quad (2.23)$$

Lemmas 2.1 and 2.2 yield all those cocycles of $H_{\text{gh}}(s_{\text{gh}})$ which do not depend either on ψ_1 and χ_1 or on $\bar{\psi}^1$ and $\bar{\chi}^1$. The following lemma provides those cocycles which are at least linear in ψ_1 or χ_1 and in $\bar{\psi}^1$ or $\bar{\chi}^1$.

Lemma 2.3.

(i) Any cocycle in $H_{\text{gh}}(s_{\text{gh}})$ which is at least linear in ψ_1 or χ_1 and in $\bar{\psi}^1$ or $\bar{\chi}^1$ is equivalent to $\tilde{c}^1\chi_1\bar{\chi}^1 - \tilde{c}^3\chi_1\bar{\psi}^1 + \tilde{c}^2\psi_1\bar{\psi}^1 - \tilde{c}^4\psi_1\bar{\chi}^1$ times a complex number:

$$\begin{aligned} s_{\text{gh}}\omega = 0, \quad \omega \in \Omega_{\text{gh}}, \quad \omega|_{\psi_1=\chi_1=0} = 0, \quad \omega|_{\bar{\psi}^1=\bar{\chi}^1=0} = 0 \quad \Rightarrow \\ \omega \sim (\tilde{c}^1\chi_1\bar{\chi}^1 - \tilde{c}^3\chi_1\bar{\psi}^1 + \tilde{c}^2\psi_1\bar{\psi}^1 - \tilde{c}^4\psi_1\bar{\chi}^1)b, \quad b \in \mathbb{C}. \end{aligned} \quad (2.24)$$

(ii) The cocycle $\tilde{c}^1\chi_1\bar{\chi}^1 - \tilde{c}^3\chi_1\bar{\psi}^1 + \tilde{c}^2\psi_1\bar{\psi}^1 - \tilde{c}^4\psi_1\bar{\chi}^1$ is nontrivial in $H_{\text{gh}}(s_{\text{gh}})$:

$$\tilde{c}^1\chi_1\bar{\chi}^1 - \tilde{c}^3\chi_1\bar{\psi}^1 + \tilde{c}^2\psi_1\bar{\psi}^1 - \tilde{c}^4\psi_1\bar{\chi}^1 \not\sim 0. \quad (2.25)$$

Proof: We expand $\omega \in \Omega_{\text{gh}}$ in ψ_1 according to

$$\omega = \sum_{m=0}^{\bar{m}} (\psi_1)^m \omega_m(\chi_1, \bar{\psi}^1, \bar{\chi}^1, \tilde{c}^1, \dots, \tilde{c}^4). \quad (2.26)$$

$\omega|_{\psi_1=\chi_1=0} = 0$ and $\omega|_{\bar{\psi}^1=\bar{\chi}^1=0} = 0$ imply that in this expansion the ω_m are polynomials in $\chi_1, \bar{\psi}^1, \bar{\chi}^1, \tilde{c}^1, \dots, \tilde{c}^4$ that are at least linear in $\bar{\psi}^1$ or $\bar{\chi}^1$ and that ω_0 is at least linear in χ_1 :

$$m > 0 : \omega_m = \bar{\psi}^1 \omega_{m,1}(\chi_1, \bar{\psi}^1, \bar{\chi}^1, \tilde{c}^1, \dots, \tilde{c}^4) + \bar{\chi}^1 \omega_{m,2}(\chi_1, \bar{\chi}^1, \tilde{c}^1, \dots, \tilde{c}^4); \quad (2.27)$$

$$\omega_0 = \chi_1 [\bar{\psi}^1 \omega_{0,1}(\chi_1, \bar{\psi}^1, \bar{\chi}^1, \tilde{c}^1, \dots, \tilde{c}^4) + \bar{\chi}^1 \omega_{0,2}(\chi_1, \bar{\chi}^1, \tilde{c}^1, \dots, \tilde{c}^4)] \quad (2.28)$$

where the arguments of $\omega_{m,1}$ and $\omega_{m,2}$ indicate that $\omega_{m,1}$ may depend polynomially on all variables $\chi_1, \bar{\psi}^1, \bar{\chi}^1, \tilde{c}^1, \dots, \tilde{c}^4$ whereas $\omega_{m,2}$ is a polynomial only in $\chi_1, \bar{\chi}^1, \tilde{c}^1, \dots, \tilde{c}^4$ but does not depend on $\bar{\psi}^1$. Notice that the expansion (2.26) is always finite since Ω_{gh} is a space of polynomials in the ghost variables; hence, there is always a term with some highest degree \bar{m} in ψ_1 . Using the decomposition (2.10) of s_{gh} , one infers that $s_{\text{gh}}\omega$ contains at most one term of degree $\bar{m} + 1$ in ψ_1 given by $(\psi_1)^{\bar{m}+1} D_1 \omega_{\bar{m}}$. Hence, $s_{\text{gh}}\omega = 0$ requires in particular that this term vanishes,

$$s_{\text{gh}}\omega = 0 \quad \Rightarrow \quad D_1 \omega_{\bar{m}} = 0. \quad (2.29)$$

$D_1 \omega_{\bar{m}} = 0$ is treated by the "basic lemma" given in [3] as follows. We introduce the antiderivation

$$r = \tilde{c}^1 \frac{\partial}{\partial \bar{\psi}^1} + \tilde{c}^3 \frac{\partial}{\partial \bar{\chi}^1}. \quad (2.30)$$

The anticommutator of r and D_1 is

$$\{r, D_1\} = L, \quad L = N_{\bar{\psi}^1} + N_{\bar{\chi}^1} + N_{\tilde{c}^1} + N_{\tilde{c}^3} \quad (2.31)$$

with $N_{\bar{\psi}^1}, N_{\bar{\chi}^1}, N_{\tilde{c}^1}, N_{\tilde{c}^3}$ defined analogously to (2.9). $\omega_{\bar{m}}$ is decomposed into eigenfunctions of L . We denote the corresponding eigenvalues by λ ; these eigenvalues are positive integers since all terms in $\omega_{\bar{m}}$ are at least linear in $\bar{\psi}^1$ or $\bar{\chi}^1$ owing to (2.27), (2.28):

$$\omega_{\bar{m}} = \sum_{\lambda \geq 1} \omega_{\bar{m},\lambda}, \quad L \omega_{\bar{m},\lambda} = \lambda \omega_{\bar{m},\lambda}, \quad \lambda \in \mathbb{N}. \quad (2.32)$$

Owing to $[L, D_1] = 0$ (which follows from $L = \{r, D_1\}$), $D_1 \omega_{\bar{m}} = 0$ implies $D_1 \omega_{\bar{m},\lambda} = 0$ for all λ ; $L = \{D_1, r\}$ then implies that all $\omega_{\bar{m},\lambda}$ are D_1 -exact:

$$\forall \lambda : \quad D_1 \omega_{\bar{m},\lambda} = 0 \quad \Rightarrow \quad \lambda \omega_{\bar{m},\lambda} = \{D_1, r\} \omega_{\bar{m},\lambda} = D_1 r \omega_{\bar{m},\lambda} \quad (2.33)$$

Hence, $\omega_{\bar{m}}$ is D_1 -exact:

$$\omega_{\bar{m}} = D_1 \sum_{\lambda \geq 1} \frac{1}{\lambda} r \omega_{\bar{m},\lambda}. \quad (2.34)$$

This implies that we can remove the term $\omega_{\bar{m}}$ of highest degree \bar{m} from ω by subtracting an s_{gh} -coboundary, if $\bar{m} > 0$:

$$\bar{m} > 0 : \omega' := \omega - s_{\text{gh}} \left((\psi_1)^{\bar{m}-1} \sum_{\lambda \geq 1} \frac{1}{\lambda} r \omega_{\bar{m},\lambda} \right)$$

$$= \sum_{m=0}^{\bar{m}-1} (\psi_1)^m \omega'_m(\chi_1, \bar{\psi}^1, \bar{\chi}^1, \tilde{c}^1, \dots, \tilde{c}^4) \quad (2.35)$$

where

$$\begin{aligned} \omega'_{\bar{m}-1} &= \omega_{\bar{m}-1} - \chi_1 D_2 \sum_{\lambda \geq 1} \frac{1}{\lambda} r \omega_{\bar{m}, \lambda}, \\ m \leq \bar{m} - 2 : \omega'_m &= \omega_m. \end{aligned} \quad (2.36)$$

It should be noted that equations (2.27), (2.28) also apply to $\omega'_{\bar{m}-1}$ because D_2 consists of contributions that are linear in $\bar{\psi}^1$ or $\bar{\chi}^1$, cf. (2.11). Repeating the above procedure for ω' and continuing it, one removes successively all terms depending on ψ_1 by subtracting s_{gh} -coboundaries. This shows that ω is s_{gh} -exact except (possibly) for a contribution $\omega'_0(\chi_1, \bar{\psi}^1, \bar{\chi}^1, \tilde{c}^1, \dots, \tilde{c}^4)$,

$$\omega \sim \omega'_0(\chi_1, \bar{\psi}^1, \bar{\chi}^1, \tilde{c}^1, \dots, \tilde{c}^4). \quad (2.37)$$

We now expand ω'_0 in χ_1 ; the coefficient functions of this expansion are in Ω^- defined in (2.5):

$$\omega'_0(\chi_1, \bar{\psi}^1, \bar{\chi}^1, \tilde{c}^1, \dots, \tilde{c}^4) = \sum_{k \geq 1} (\chi_1)^k \hat{\omega}_k, \quad \hat{\omega}_k \in \Omega^-. \quad (2.38)$$

This yields

$$s_{\text{gh}} \omega'_0 = \sum_{k \geq 1} (\psi_1 (\chi_1)^k D_1 \hat{\omega}_k + (\chi_1)^{k+1} D_2 \hat{\omega}_k)$$

and thus

$$s_{\text{gh}} \omega'_0 = 0 \Leftrightarrow \forall k : D_1 \hat{\omega}_k = 0 \wedge D_2 \hat{\omega}_k = 0 \Leftrightarrow \forall k : s_{\text{gh}} \hat{\omega}_k = 0. \quad (2.39)$$

Using now the result (2.6) and that ω'_0 takes the form (2.28), we obtain

$$\begin{aligned} \hat{\omega}_k &= \bar{\psi}^1 a_k(\bar{\psi}^1, \bar{\chi}^1) + \bar{\chi}^1 b_k(\bar{\chi}^1) + (\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) c_k(\bar{\psi}^1, \bar{\chi}^1) \\ &\quad + (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) d_k(\bar{\psi}^1, \bar{\chi}^1) + (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) (\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) e_k(\bar{\psi}^1, \bar{\chi}^1) \end{aligned} \quad (2.40)$$

for some polynomials a_k, c_k, d_k, e_k in $\bar{\psi}^1$ and $\bar{\chi}^1$ and some polynomials b_k in $\bar{\chi}^1$. Since in ω'_0 the various terms in (2.40) are multiplied by at least one power of χ_1 , cf. (2.38), all of them provide s_{gh} -exact contributions to ω'_0 except for $d_1(\bar{\psi}^1, \bar{\chi}^1)$ because one has:

$$\begin{aligned} \chi_1 \bar{\psi}^1 &= s_{\text{gh}} \tilde{c}^4, \quad \chi_1 \bar{\chi}^1 = s_{\text{gh}} \tilde{c}^2, \quad \chi_1 (\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) = s_{\text{gh}} (\tilde{c}^4 \tilde{c}^2), \\ (\chi_1)^2 (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) &= s_{\text{gh}} (\tilde{c}^3 \tilde{c}^4 \chi_1 - \tilde{c}^1 \tilde{c}^2 \chi_1 - \tilde{c}^2 \tilde{c}^4 \psi_1), \\ \chi_1 (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) (\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) &= s_{\text{gh}} [(\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) \tilde{c}^2 \tilde{c}^4]. \end{aligned} \quad (2.41)$$

The contributions to $d_1(\bar{\psi}^1, \bar{\chi}^1)$ which are at least linear in $\bar{\psi}^1$ or $\bar{\chi}^1$ also provide only s_{gh} -exact contributions to ω'_0 owing to:

$$\chi_1 \bar{\psi}^1 (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) = s_{\text{gh}} [\tilde{c}^4 (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1)],$$

$$\chi_1 \bar{\chi}^1 (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) = s_{\text{gh}} [\tilde{c}^2 (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1)]. \quad (2.42)$$

The only part of $d_1(\bar{\psi}^1, \bar{\chi}^1)$ which provides a possibly nontrivial contribution to ω'_0 is thus the part which does not depend on $\bar{\psi}^1$ and $\bar{\chi}^1$ at all. We denote this part by $2b \in \mathbb{C}$ and write the corresponding contribution to ω'_0 as:

$$\begin{aligned} 2b \chi_1 (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) &= b (\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1) \\ &\quad + b (\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 - \tilde{c}^2 \psi_1 \bar{\psi}^1 + \tilde{c}^4 \psi_1 \bar{\chi}^1) \\ &= b (\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1) \\ &\quad + b s_{\text{gh}} (\tilde{c}^3 \tilde{c}^4 - \tilde{c}^1 \tilde{c}^2). \end{aligned} \quad (2.43)$$

We conclude

$$\omega'_0(\chi_1, \bar{\psi}^1, \bar{\chi}^1, \tilde{c}^1, \dots, \tilde{c}^4) \sim (\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1) b. \quad (2.44)$$

Together with (2.37) this yields part (i) of lemma 2.3.

To prove that the cocycle $\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1$ is no coboundary in $H_{\text{gh}}(s_{\text{gh}})$, we use arguments as in the paragraph preceding lemma 3.1 in [2]: in order to be a coboundary, $\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1$ would have to be of the form $s_{\text{gh}}(d_{ab} c^a c^b)$ for some $d_{ab} \in \mathbb{C}$ but no such d_{ab} exist. The non-existence of d_{ab} can be inferred without any computation from the fact that $\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1$ is actually an $\mathfrak{so}(t, 4-t)$ -invariant ghost polynomial, cf. section 2.2, and therefore, owing to the $\mathfrak{so}(t, 4-t)$ -invariance of s_{gh} , $d_{ab} c^a c^b$ would have to be $\mathfrak{so}(t, 4-t)$ -invariant too; however, there is no nonvanishing $\mathfrak{so}(t, 4-t)$ -invariant bilinear polynomial in the translation ghosts (the only candidate bilinear polynomial would be proportional to $\eta_{ab} c^a c^b$ but this vanishes as the translation ghosts anticommute). This yields part (ii) of lemma 2.3 and completes the proof of the lemma. \blacksquare

Comment: Equations (2.43) show that the cocycle $\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1$ is equivalent to the seemingly simpler cocycle $2\chi_1(\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1)$. Nevertheless we prefer to use the cocycle $\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1$ owing to its $\mathfrak{so}(t, 4-t)$ -invariance.

Lemmas 2.1, 2.2 and 2.3 provide the cohomology $H_{\text{gh}}(s_{\text{gh}})$ in the spinor representations (1.5) because the various nontrivial cocycles in these lemmas cannot combine to coboundaries. The latter statement holds because these cocycles have different degrees in ψ_1 and χ_1 or $\bar{\psi}^1$ and $\bar{\chi}^1$ respectively, while s_{gh} increments both of these degrees by one unit. We thus conclude:

Lemma 2.4 ($H_{\text{gh}}(s_{\text{gh}})$ for $N = 1$).

(i) In the spinor representations (1.5) any cocycle $\omega \in \Omega_{\text{gh}}$ is equivalent to a linear combination of a polynomial in $\bar{\psi}^1, \bar{\chi}^1, \tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1$ and $\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1$, of a polynomial in $\psi_1, \chi_1, \tilde{c}^2 \psi_1 - \tilde{c}^3 \chi_1$ and $\tilde{c}^1 \chi_1 - \tilde{c}^4 \psi_1$, and of $\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1$:

$$\begin{aligned} \omega \in \Omega_{\text{gh}} : \quad s_{\text{gh}} \omega = 0 \quad \Leftrightarrow \quad &\omega \sim p_1(\bar{\psi}^1, \bar{\chi}^1) + q_1(\psi_1, \chi_1) \\ &+ (\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) p_2(\bar{\psi}^1, \bar{\chi}^1) \end{aligned}$$

$$\begin{aligned}
& + (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) p_3(\bar{\psi}^1, \bar{\chi}^1) \\
& + (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1)(\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) p_4(\bar{\psi}^1, \bar{\chi}^1) \\
& + (\tilde{c}^2 \psi_1 - \tilde{c}^3 \chi_1) q_2(\psi_1, \chi_1) \\
& + (\tilde{c}^1 \chi_1 - \tilde{c}^4 \psi_1) q_3(\psi_1, \chi_1) \\
& + (\tilde{c}^1 \chi_1 - \tilde{c}^4 \psi_1)(\tilde{c}^2 \psi_1 - \tilde{c}^3 \chi_1) q_4(\psi_1, \chi_1) \\
& + (\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1) b \quad (2.45)
\end{aligned}$$

with polynomials $p_1(\bar{\psi}^1, \bar{\chi}^1), \dots, p_4(\bar{\psi}^1, \bar{\chi}^1)$ in $\bar{\psi}^1$ and $\bar{\chi}^1$, polynomials $q_1(\psi_1, \chi_1), \dots, q_4(\psi_1, \chi_1)$ in ψ_1 and χ_1 , and $b \in \mathbb{C}$.

(ii) A linear combination of a polynomial in $\bar{\psi}^1, \bar{\chi}^1, \tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1$ and $\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1$, of a polynomial in $\psi_1, \chi_1, \tilde{c}^2 \psi_1 - \tilde{c}^3 \chi_1$ and $\tilde{c}^1 \chi_1 - \tilde{c}^4 \psi_1$, and of $\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1$ is exact in $H_{\text{gh}}(s_{\text{gh}})$ if and only if it vanishes:

$$\begin{aligned}
0 & \sim p_1(\bar{\psi}^1, \bar{\chi}^1) + q_1(\psi_1, \chi_1) \\
& + (\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) p_2(\bar{\psi}^1, \bar{\chi}^1) \\
& + (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) p_3(\bar{\psi}^1, \bar{\chi}^1) \\
& + (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1)(\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) p_4(\bar{\psi}^1, \bar{\chi}^1) \\
& + (\tilde{c}^2 \psi_1 - \tilde{c}^3 \chi_1) q_2(\psi_1, \chi_1) \\
& + (\tilde{c}^1 \chi_1 - \tilde{c}^4 \psi_1) q_3(\psi_1, \chi_1) \\
& + (\tilde{c}^1 \chi_1 - \tilde{c}^4 \psi_1)(\tilde{c}^2 \psi_1 - \tilde{c}^3 \chi_1) q_4(\psi_1, \chi_1) \\
& + (\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1) b \\
\Leftrightarrow & p_1(\bar{\psi}^1, \bar{\chi}^1) + q_1(\psi_1, \chi_1) = 0 \wedge b = 0 \wedge \\
& \forall i \in \{2, 3, 4\} : p_i(\bar{\psi}^1, \bar{\chi}^1) = 0 = q_i(\psi_1, \chi_1). \quad (2.46)
\end{aligned}$$

Comment: Notice that in (2.46) the condition $p_1(\bar{\psi}^1, \bar{\chi}^1) + q_1(\psi_1, \chi_1) = 0$ imposes that $p_1(\bar{\psi}^1, \bar{\chi}^1)$ and $q_1(\psi_1, \chi_1)$ do not depend on supersymmetry ghosts at all,

$$p_1(\bar{\psi}^1, \bar{\chi}^1) + q_1(\psi_1, \chi_1) = 0 \Leftrightarrow p_1, q_1 \in \mathbb{C} \wedge p_1 = -q_1. \quad (2.47)$$

2.1.2 $H_{\text{gh}}(s_{\text{gh}})$ for $N = 2$

We shall first prove the following lemma:

Lemma 2.5. *The general solution of the cocycle condition in $H_{\text{gh}}(s_{\text{gh}})$ for $N = 2$ in the spinor representations (1.5) is:*

$$\begin{aligned}
s_{\text{gh}} \omega = 0 & \Leftrightarrow \omega \sim p(\bar{\psi}^1, \bar{\chi}^1, \xi_2) + q(\psi_1, \chi_1, \xi_2) \\
& + (-\tilde{c}^2 \bar{\psi}^1 \psi_2 + \tilde{c}^4 \bar{\chi}^1 \psi_2 - \tilde{c}^1 \bar{\chi}^1 \chi_2 + \tilde{c}^3 \bar{\psi}^1 \chi_2) h(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\
& + (-\tilde{c}^2 \psi_1 \bar{\psi}^2 + \tilde{c}^3 \chi_1 \bar{\psi}^2 - \tilde{c}^1 \chi_1 \bar{\chi}^2 + \tilde{c}^4 \psi_1 \bar{\chi}^2) g(\psi_1, \chi_1, \xi_2) \\
& + (\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1 \\
& - \tilde{c}^1 \chi_2 \bar{\chi}^2 + \tilde{c}^3 \chi_2 \bar{\psi}^2 - \tilde{c}^2 \psi_2 \bar{\psi}^2 + \tilde{c}^4 \psi_2 \bar{\chi}^2) b(\xi_2) \quad (2.48)
\end{aligned}$$

with arbitrary polynomials $p(\bar{\psi}^1, \bar{\chi}^1, \xi_2), h(\bar{\psi}^1, \bar{\chi}^1, \xi_2)$ in $\bar{\psi}^1, \bar{\chi}^1, \psi_2, \chi_2, \bar{\psi}^2, \bar{\chi}^2$, arbitrary polynomials $q(\psi_1, \chi_1, \xi_2), g(\psi_1, \chi_1, \xi_2)$ in $\psi_1, \chi_1, \psi_2, \chi_2, \bar{\psi}^2, \bar{\chi}^2$, and an arbitrary polynomial $b(\xi_2)$ in $\psi_2, \chi_2, \bar{\psi}^2, \bar{\chi}^2$.

Proof: We split the coboundary operator s_{gh} according to

$$s_{\text{gh}} = s_{\text{gh},1} + s_{\text{gh},2} \quad (2.49)$$

into a first operator $s_{\text{gh},1}$ which increments the degree in the supersymmetry ghosts ξ_1^α (ξ_1 -degree) by two units and a second operator $s_{\text{gh},2}$ which increments the degree in the supersymmetry ghosts ξ_2^α (ξ_2 -degree) by two units,

$$\begin{aligned} s_{\text{gh},1} &= \psi_1 \bar{\psi}^1 \frac{\partial}{\partial \tilde{c}^1} + \chi_1 \bar{\chi}^1 \frac{\partial}{\partial \tilde{c}^2} + \psi_1 \bar{\chi}^1 \frac{\partial}{\partial \tilde{c}^3} + \chi_1 \bar{\psi}^1 \frac{\partial}{\partial \tilde{c}^4}, \\ s_{\text{gh},2} &= \psi_2 \bar{\psi}^2 \frac{\partial}{\partial \tilde{c}^1} + \chi_2 \bar{\chi}^2 \frac{\partial}{\partial \tilde{c}^2} + \psi_2 \bar{\chi}^2 \frac{\partial}{\partial \tilde{c}^3} + \chi_2 \bar{\psi}^2 \frac{\partial}{\partial \tilde{c}^4}. \end{aligned} \quad (2.50)$$

We denote by N_{ξ_1} and N_{ξ_2} the counting operators which measure the ξ_1 -degree and ξ_2 -degree respectively:

$$\begin{aligned} N_{\xi_1} &= \xi_1^\alpha \frac{\partial}{\partial \xi_1^\alpha} = \psi_1 \frac{\partial}{\partial \psi_1} + \chi_1 \frac{\partial}{\partial \chi_1} + \bar{\psi}^1 \frac{\partial}{\partial \bar{\psi}^1} + \bar{\chi}^1 \frac{\partial}{\partial \bar{\chi}^1}, \\ N_{\xi_2} &= \xi_2^\alpha \frac{\partial}{\partial \xi_2^\alpha} = \psi_2 \frac{\partial}{\partial \psi_2} + \chi_2 \frac{\partial}{\partial \chi_2} + \bar{\psi}^2 \frac{\partial}{\partial \bar{\psi}^2} + \bar{\chi}^2 \frac{\partial}{\partial \bar{\chi}^2}. \end{aligned} \quad (2.51)$$

The first operator $s_{\text{gh},1}$ and the second operator $s_{\text{gh},2}$ and these counting operators fulfill the algebra

$$\begin{aligned} (s_{\text{gh},1})^2 &= \{s_{\text{gh},1}, s_{\text{gh},2}\} = (s_{\text{gh},2})^2 = 0, \\ [N_{\xi_1}, s_{\text{gh},1}] &= 2s_{\text{gh},1}, \quad [N_{\xi_1}, s_{\text{gh},2}] = 0, \\ [N_{\xi_2}, s_{\text{gh},1}] &= 0, \quad [N_{\xi_2}, s_{\text{gh},2}] = 2s_{\text{gh},2}. \end{aligned} \quad (2.52)$$

In order to determine $H_{\text{gh}}(s_{\text{gh}})$ for $N = 2$ we decompose the elements $\omega \in \Omega_{\text{gh}}$ into eigenfunctions ω_m with ξ_1 -degree m :

$$\omega = \sum_{m=0}^{\bar{m}} \omega_m, \quad N_{\xi_1} \omega_m = m \omega_m. \quad (2.53)$$

We shall now analyse the cocycle condition $s_{\text{gh}} \omega = 0$ in $H_{\text{gh}}(s_{\text{gh}})$ by decomposing it according to the ξ_1 -degree:

$$s_{\text{gh}} \omega = 0 \Leftrightarrow \begin{cases} s_{\text{gh},1} \omega_{\bar{m}} = 0, & s_{\text{gh},1} \omega_{\bar{m}-2} + s_{\text{gh},2} \omega_{\bar{m}} = 0, & \dots \\ s_{\text{gh},1} \omega_{\bar{m}-1} = 0, & s_{\text{gh},1} \omega_{\bar{m}-3} + s_{\text{gh},2} \omega_{\bar{m}-1} = 0, & \dots \end{cases} \quad (2.54)$$

where the equations in the first line contain the $\omega_{\bar{m}-2k}$ while the equations in the second line contain the $\omega_{\bar{m}-2k-1}$ ($k = 0, 1, \dots$). The equations of the two lines are independent and analogous to each other. Hence, it suffices to discuss the equations in the first line.

The cohomologies of $s_{\text{gh},1}$ and $s_{\text{gh},2}$ are known from lemma 2.4. Indeed, $s_{\text{gh},1}$ acts on polynomials in the \tilde{c}^a , ψ_i , χ_i , $\bar{\psi}^i$, $\bar{\chi}^i$ (with $a = 1, \dots, 4$ and $i = 1, 2$) exactly as s_{gh} in the case $N = 1$ on polynomials in the \tilde{c}^a , ψ_1 , χ_1 , $\bar{\psi}^1$, $\bar{\chi}^1$ with ψ_2 , χ_2 , $\bar{\psi}^2$, $\bar{\chi}^2$

treated as ordinary complex numbers. The cohomology of $s_{\text{gh},2}$ is obtained from the cohomology of $s_{\text{gh},1}$ by interchanging the roles of $\psi_1, \chi_1, \bar{\psi}^1, \bar{\chi}^1$ and $\psi_2, \chi_2, \bar{\psi}^2, \bar{\chi}^2$. Using the result (2.45) of lemma 2.4 we infer from equation $s_{\text{gh},1}\omega_{\bar{m}} = 0$ in (2.54) that

$$\begin{aligned}\omega_{\bar{m}} = & s_{\text{gh},1}\eta_{\bar{m}-2} + p_{1,\bar{m}}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) + q_{1,\bar{m}}(\psi_1, \chi_1, \xi_2) \\ & + (-\tilde{c}^2\bar{\psi}^1 + \tilde{c}^4\bar{\chi}^1)p_{2,\bar{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\ & + (\tilde{c}^1\bar{\chi}^1 - \tilde{c}^3\bar{\psi}^1)p_{3,\bar{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\ & + (\tilde{c}^1\bar{\chi}^1 - \tilde{c}^3\bar{\psi}^1)(-\tilde{c}^2\bar{\psi}^1 + \tilde{c}^4\bar{\chi}^1)p_{4,\bar{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\ & + (-\tilde{c}^2\psi_1 + \tilde{c}^3\chi_1)q_{2,\bar{m}-1}(\psi_1, \chi_1, \xi_2) \\ & + (-\tilde{c}^1\chi_1 + \tilde{c}^4\psi_1)q_{3,\bar{m}-1}(\psi_1, \chi_1, \xi_2) \\ & + (-\tilde{c}^1\chi_1 + \tilde{c}^4\psi_1)(-\tilde{c}^2\psi_1 + \tilde{c}^3\chi_1)q_{4,\bar{m}-2}(\psi_1, \chi_1, \xi_2) \\ & + \delta_{\bar{m}}^2(\tilde{c}^1\chi_1\bar{\chi}^1 - \tilde{c}^3\chi_1\bar{\psi}^1 + \tilde{c}^2\psi_1\bar{\psi}^1 - \tilde{c}^4\psi_1\bar{\chi}^1)b(\xi_2)\end{aligned}\quad (2.55)$$

for some polynomials $\eta_{\bar{m}-2}, p_{1,\bar{m}}(\bar{\psi}^1, \bar{\chi}^1, \xi_2), \dots, b(\xi_2)$ where the subscripts $\bar{m}, \bar{m}-1, \bar{m}-2$ indicate the ξ_1 -degree and the Kronecker symbol $\delta_{\bar{m}}^2$ in front of the term in the last line indicates that this term contributes only in the case $\bar{m} = 2$.

We proceed to the equation $s_{\text{gh},1}\omega_{\bar{m}-2} + s_{\text{gh},2}\omega_{\bar{m}} = 0$ in (2.54) and use there the result (2.55) for $\omega_{\bar{m}}$. By a straightforward computation this yields:

$$\begin{aligned}0 = & s_{\text{gh},1}(\omega_{\bar{m}-2} - s_{\text{gh},2}\eta_{\bar{m}-2}) \\ & + (\bar{\psi}^2\bar{\chi}^1 - \bar{\chi}^2\bar{\psi}^1)(\chi_2 p_{2,\bar{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) + \psi_2 p_{3,\bar{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2)) \\ & + (\psi_2\chi_1 - \chi_2\psi_1)(\bar{\chi}^2 q_{2,\bar{m}-1}(\psi_1, \chi_1, \xi_2) - \bar{\psi}^2 q_{3,\bar{m}-1}(\psi_1, \chi_1, \xi_2)) \\ & + (-\tilde{c}^2\bar{\psi}^1 + \tilde{c}^4\bar{\chi}^1)(\bar{\psi}^2\bar{\chi}^1 - \bar{\chi}^2\bar{\psi}^1)\psi_2 p_{4,\bar{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\ & - (\tilde{c}^1\bar{\chi}^1 - \tilde{c}^3\bar{\psi}^1)(\bar{\psi}^2\bar{\chi}^1 - \bar{\chi}^2\bar{\psi}^1)\chi_2 p_{4,\bar{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\ & - (-\tilde{c}^2\psi_1 + \tilde{c}^3\chi_1)(\psi_2\chi_1 - \chi_2\psi_1)\bar{\psi}^2 q_{4,\bar{m}-2}(\psi_1, \chi_1, \xi_2) \\ & - (-\tilde{c}^1\chi_1 + \tilde{c}^4\psi_1)(\psi_2\chi_1 - \chi_2\psi_1)\bar{\chi}^2 q_{4,\bar{m}-2}(\psi_1, \chi_1, \xi_2) \\ & + \delta_{\bar{m}}^2 s_{\text{gh},1}(\tilde{c}^1\chi_2\bar{\chi}^2 - \tilde{c}^3\chi_2\bar{\psi}^2 + \tilde{c}^2\psi_2\bar{\psi}^2 - \tilde{c}^4\psi_2\bar{\chi}^2)b(\xi_2).\end{aligned}\quad (2.56)$$

The terms in the first line and in the last line of equation (2.56) are $s_{\text{gh},1}$ -exact. Hence, the sum of the other terms is $s_{\text{gh},1}$ -exact. Using the result (2.46) of lemma 2.4, we infer:

$$\chi_2 p_{2,\bar{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) + \psi_2 p_{3,\bar{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) = 0, \quad (2.57)$$

$$\bar{\chi}^2 q_{2,\bar{m}-1}(\psi_1, \chi_1, \xi_2) - \bar{\psi}^2 q_{3,\bar{m}-1}(\psi_1, \chi_1, \xi_2) = 0, \quad (2.58)$$

$$p_{4,\bar{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) = 0, \quad q_{4,\bar{m}-2}(\psi_1, \chi_1, \xi_2) = 0. \quad (2.59)$$

Equations (2.57) and (2.58) imply:

$$\begin{aligned}p_{2,\bar{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) &= \psi_2 h_{\bar{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2), \\ p_{3,\bar{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) &= -\chi_2 h_{\bar{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2), \\ q_{2,\bar{m}-1}(\psi_1, \chi_1, \xi_2) &= \bar{\psi}^2 g_{\bar{m}-1}(\psi_1, \chi_1, \xi_2),\end{aligned}$$

$$q_{3,\overline{m}-1}(\psi_1, \chi_1, \xi_2) = \bar{\chi}^2 g_{\overline{m}-1}(\psi_1, \chi_1, \xi_2) \quad (2.60)$$

for some polynomials $h_{\overline{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2)$ and $g_{\overline{m}-1}(\psi_1, \chi_1, \xi_2)$. Using the results (2.59) and (2.60) in equation (2.55), the latter gives:

$$\begin{aligned} \omega_{\overline{m}} &= s_{\text{gh},1} \eta_{\overline{m}-2} + p_{1,\overline{m}}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) + q_{1,\overline{m}}(\psi_1, \chi_1, \xi_2) \\ &\quad + (-\tilde{c}^2 \bar{\psi}^1 \psi_2 + \tilde{c}^4 \bar{\chi}^1 \psi_2 - \tilde{c}^1 \bar{\chi}^1 \chi_2 + \tilde{c}^3 \bar{\psi}^1 \chi_2) h_{\overline{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\ &\quad + (-\tilde{c}^2 \psi_1 \bar{\psi}^2 + \tilde{c}^3 \chi_1 \bar{\psi}^2 - \tilde{c}^1 \chi_1 \bar{\chi}^2 + \tilde{c}^4 \psi_1 \bar{\chi}^2) g_{\overline{m}-1}(\psi_1, \chi_1, \xi_2) \\ &\quad + \delta_{\overline{m}}^2 (\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1) b(\xi_2). \end{aligned} \quad (2.61)$$

Using the results (2.57) to (2.60) in equation (2.56), the latter becomes

$$0 = s_{\text{gh},1} [\omega_{\overline{m}-2} - s_{\text{gh},2} \eta_{\overline{m}-2} + \delta_{\overline{m}}^2 (\tilde{c}^1 \chi_2 \bar{\chi}^2 - \tilde{c}^3 \chi_2 \bar{\psi}^2 + \tilde{c}^2 \psi_2 \bar{\psi}^2 - \tilde{c}^4 \psi_2 \bar{\chi}^2) b(\xi_2)]. \quad (2.62)$$

This is an equation like $s_{\text{gh},1} \omega_{\overline{m}} = 0$ in (2.54), with $\omega_{\overline{m}-2} - s_{\text{gh},2} \eta_{\overline{m}-2} + \delta_{\overline{m}}^2 (\tilde{c}^1 \chi_2 \bar{\chi}^2 - \tilde{c}^3 \chi_2 \bar{\psi}^2 + \tilde{c}^2 \psi_2 \bar{\psi}^2 - \tilde{c}^4 \psi_2 \bar{\chi}^2) b(\xi_2)$ in place of $\omega_{\overline{m}}$. One can analyse this equation as $s_{\text{gh},1} \omega_{\overline{m}} = 0$ above and obtains that $\omega_{\overline{m}-2}$ is given by terms as in equation (2.61) plus the $s_{\text{gh},2}$ -coboundary $s_{\text{gh},2} \eta_{\overline{m}-2}$ and the term $(\tilde{c}^1 \chi_2 \bar{\chi}^2 - \tilde{c}^3 \chi_2 \bar{\psi}^2 + \tilde{c}^2 \psi_2 \bar{\psi}^2 - \tilde{c}^4 \psi_2 \bar{\chi}^2) b(\xi_2)$ in the case $\overline{m} = 2$. Proceeding analogously to terms of lower ξ_1 -degree, one obtains lemma 2.5. \blacksquare

To completely characterize $H_{\text{gh}}(s_{\text{gh}})$ for $N = 2$ in the spinor representation (1.5), we still have to determine those cocycles occurring in (2.48) that are coboundaries in $H_{\text{gh}}(s_{\text{gh}})$. In other words: we still have to determine those ghost polynomials $p(\bar{\psi}^1, \bar{\chi}^1, \xi_2)$, $q(\psi_1, \chi_1, \xi_2)$, $h(\bar{\psi}^1, \bar{\chi}^1, \xi_2)$, $g(\psi_1, \chi_1, \xi_2)$ and $b(\xi_2)$ for which the cocycles given in (2.48) combine to a coboundary in $H_{\text{gh}}(s_{\text{gh}})$. The solution to this problem is the following lemma 2.6 which together with lemma 2.5 provides an exhaustive characterization of $H_{\text{gh}}(s_{\text{gh}})$ for $N = 2$ in the spinor representations (1.5).

Lemma 2.6 (Coboundaries in lemma 2.5).

$$\begin{aligned} 0 &\sim p(\bar{\psi}^1, \bar{\chi}^1, \xi_2) + q(\psi_1, \chi_1, \xi_2) \\ &\quad + (-\tilde{c}^2 \bar{\psi}^1 \psi_2 + \tilde{c}^4 \bar{\chi}^1 \psi_2 - \tilde{c}^1 \bar{\chi}^1 \chi_2 + \tilde{c}^3 \bar{\psi}^1 \chi_2) h(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\ &\quad + (-\tilde{c}^2 \psi_1 \bar{\psi}^2 + \tilde{c}^3 \chi_1 \bar{\psi}^2 - \tilde{c}^1 \chi_1 \bar{\chi}^2 + \tilde{c}^4 \psi_1 \bar{\chi}^2) g(\psi_1, \chi_1, \xi_2) \\ &\quad + (\tilde{c}^1 \chi_1 \bar{\chi}^1 - \tilde{c}^3 \chi_1 \bar{\psi}^1 + \tilde{c}^2 \psi_1 \bar{\psi}^1 - \tilde{c}^4 \psi_1 \bar{\chi}^1 \\ &\quad \quad - \tilde{c}^1 \chi_2 \bar{\chi}^2 + \tilde{c}^3 \chi_2 \bar{\psi}^2 - \tilde{c}^2 \psi_2 \bar{\psi}^2 + \tilde{c}^4 \psi_2 \bar{\chi}^2) b(\xi_2) \\ \Leftrightarrow &\quad p(\bar{\psi}^1, \bar{\chi}^1, \xi_2) + q(\psi_1, \chi_1, \xi_2) \\ &\quad = (\bar{\psi}^2 \bar{\chi}^1 - \bar{\chi}^2 \bar{\psi}^1) (\chi_2 \tilde{p}_2(\bar{\psi}^1, \bar{\chi}^1, \xi_2) + \psi_2 \tilde{p}_3(\bar{\psi}^1, \bar{\chi}^1, \xi_2)) \\ &\quad \quad + (\psi_2 \chi_1 - \chi_2 \psi_1) (\bar{\chi}^2 \tilde{q}_2(\psi_1, \chi_1, \xi_2) - \bar{\psi}^2 \tilde{q}_3(\psi_1, \chi_1, \xi_2)), \\ &\quad h(\bar{\psi}^1, \bar{\chi}^1, \xi_2) = (\bar{\psi}^2 \bar{\chi}^1 - \bar{\chi}^2 \bar{\psi}^1) \tilde{p}_4(\bar{\psi}^1, \bar{\chi}^1, \xi_2), \\ &\quad g(\psi_1, \chi_1, \xi_2) = -(\psi_2 \chi_1 - \chi_2 \psi_1) \tilde{q}_4(\psi_1, \chi_1, \xi_2), \\ &\quad b(\xi_2) = 0 \end{aligned} \quad (2.63)$$

for polynomials $\tilde{p}_2(\bar{\psi}^1, \bar{\chi}^1, \xi_2)$, $\tilde{p}_3(\bar{\psi}^1, \bar{\chi}^1, \xi_2)$, $\tilde{p}_4(\bar{\psi}^1, \bar{\chi}^1, \xi_2)$ in $\bar{\psi}^1, \bar{\chi}^1, \psi_2, \chi_2, \bar{\psi}^2, \bar{\chi}^2$, and polynomials $\tilde{q}_2(\psi_1, \chi_1, \xi_2)$, $\tilde{q}_3(\psi_1, \chi_1, \xi_2)$, $\tilde{q}_4(\psi_1, \chi_1, \xi_2)$ in $\psi_1, \chi_1, \psi_2, \chi_2, \bar{\psi}^2, \bar{\chi}^2$.

Proof: We study the coboundary condition

$$\begin{aligned}
\omega = s_{\text{gh}}\eta, \quad \omega = & p(\bar{\psi}^1, \bar{\chi}^1, \xi_2) + q(\psi_1, \chi_1, \xi_2) \\
& + (-\tilde{c}^2\bar{\psi}^1\psi_2 + \tilde{c}^4\bar{\chi}^1\psi_2 - \tilde{c}^1\bar{\chi}^1\chi_2 + \tilde{c}^3\bar{\psi}^1\chi_2)h(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\
& + (-\tilde{c}^2\psi_1\bar{\psi}^2 + \tilde{c}^3\chi_1\bar{\psi}^2 - \tilde{c}^1\chi_1\bar{\chi}^2 + \tilde{c}^4\psi_1\bar{\chi}^2)g(\psi_1, \chi_1, \xi_2) \\
& + (\tilde{c}^1\chi_1\bar{\chi}^1 - \tilde{c}^3\chi_1\bar{\psi}^1 + \tilde{c}^2\psi_1\bar{\psi}^1 - \tilde{c}^4\psi_1\bar{\chi}^1 \\
& - \tilde{c}^1\chi_2\bar{\chi}^2 + \tilde{c}^3\chi_2\bar{\psi}^2 - \tilde{c}^2\psi_2\bar{\psi}^2 + \tilde{c}^4\psi_2\bar{\chi}^2)b(\xi_2).
\end{aligned} \tag{2.64}$$

Again, we use a decomposition according to the ξ_1 -degree:

$$\begin{aligned}
\omega &= \sum_{m=0}^{\bar{m}} \omega_m, \quad N_{\xi_1}\omega_m = m\omega_m, \\
\eta &= \sum_{m=0}^{\bar{p}} \eta_m, \quad N_{\xi_1}\eta_m = m\eta_m
\end{aligned} \tag{2.65}$$

where $\omega_{\bar{m}}$ and $\eta_{\bar{p}}$ do not vanish, respectively.

If $\bar{p} > \bar{m}$, the coboundary condition (2.64) yields $s_{\text{gh},1}\eta_{\bar{p}} = 0$ and $s_{\text{gh},1}\eta_{\bar{p}-2} + s_{\text{gh},2}\eta_{\bar{p}} = 0$ at ξ_1 -degrees $\bar{p}+2$ and \bar{p} , respectively. These equations imply by the same analysis as in the proof of lemma 2.5 that $\eta_{\bar{p}}$ is of the form given in (2.61). Since contributions to η of that form provide cocycles in $H_{\text{gh}}(s_{\text{gh}})$, they do not contribute to the coboundary condition (2.64) and are thus irrelevant to this coboundary condition. Hence, with no loss of generality we can assume $\bar{p} \leq \bar{m}$.

If $\bar{p} < \bar{m} - 2$, the coboundary condition (2.64) yields $\omega_{\bar{m}} = 0$ at ξ_1 -degree \bar{m} which contradicts that $\omega_{\bar{m}}$ does not vanish.

If $\bar{p} = \bar{m} - 2$ or $\bar{p} = \bar{m} - 1$ the coboundary condition (2.64) yields $\omega_{\bar{m}} = s_{\text{gh},1}\eta_{\bar{m}-2}$ at ξ_1 -degree \bar{m} , i.e., $\omega_{\bar{m}}$ is $s_{\text{gh},1}$ -exact. This implies $\omega_{\bar{m}} = 0$ by the result (2.46) in lemma 2.4 and also contradicts that $\omega_{\bar{m}}$ does not vanish.

Hence, with no loss of generality we can assume $\bar{p} = \bar{m}$ which yields the following decomposition of the coboundary condition (2.64):

$$s_{\text{gh},1}\eta_{\bar{m}} = 0, \quad s_{\text{gh},1}\eta_{\bar{m}-1} = 0, \tag{2.66}$$

$$\omega_m = s_{\text{gh},1}\eta_{m-2} + s_{\text{gh},2}\eta_m \quad \text{for } m = 2, \dots, \bar{m}, \tag{2.67}$$

$$\omega_1 = s_{\text{gh},2}\eta_1, \quad \omega_0 = s_{\text{gh},2}\eta_0. \tag{2.68}$$

From the first equation (2.66) we infer by the same arguments that led to equation (2.55):

$$\begin{aligned}
\eta_{\bar{m}} = & s_{\text{gh},1}\tilde{\eta}_{\bar{m}-2} + \tilde{p}_{1,\bar{m}}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) + \tilde{q}_{1,\bar{m}}(\psi_1, \chi_1, \xi_2) \\
& + (-\tilde{c}^2\bar{\psi}^1 + \tilde{c}^4\bar{\chi}^1)\tilde{p}_{2,\bar{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\
& + (\tilde{c}^1\bar{\chi}^1 - \tilde{c}^3\bar{\psi}^1)\tilde{p}_{3,\bar{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\
& + (\tilde{c}^1\bar{\chi}^1 - \tilde{c}^3\bar{\psi}^1)(-\tilde{c}^2\bar{\psi}^1 + \tilde{c}^4\bar{\chi}^1)\tilde{p}_{4,\bar{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\
& + (-\tilde{c}^2\psi_1 + \tilde{c}^3\chi_1)\tilde{q}_{2,\bar{m}-1}(\psi_1, \chi_1, \xi_2)
\end{aligned}$$

$$\begin{aligned}
& + (-\tilde{c}^1\chi_1 + \tilde{c}^4\psi_1)\tilde{q}_{3,\overline{m}-1}(\psi_1, \chi_1, \xi_2) \\
& + (-\tilde{c}^1\chi_1 + \tilde{c}^4\psi_1)(-\tilde{c}^2\psi_1 + \tilde{c}^3\chi_1)\tilde{q}_{4,\overline{m}-2}(\psi_1, \chi_1, \xi_2) \\
& + \delta_{\overline{m}}^2(\tilde{c}^1\chi_1\bar{\chi}^1 - \tilde{c}^3\chi_1\bar{\psi}^1 + \tilde{c}^2\psi_1\bar{\psi}^1 - \tilde{c}^4\psi_1\bar{\chi}^1)\tilde{b}(\xi_2).
\end{aligned} \tag{2.69}$$

The second equation (2.66) implies an analogous result for $\eta_{\overline{m}-1}$.

Using the result (2.69) in the equation (2.67) for $m = \overline{m}$, we obtain

$$\begin{aligned}
\omega_{\overline{m}} = & (\bar{\psi}^2\bar{\chi}^1 - \bar{\chi}^2\bar{\psi}^1)(\chi_2\tilde{p}_{2,\overline{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) + \psi_2\tilde{p}_{3,\overline{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2)) \\
& + (\psi_2\chi_1 - \chi_2\psi_1)(\bar{\chi}^2\tilde{q}_{2,\overline{m}-1}(\psi_1, \chi_1, \xi_2) - \bar{\psi}^2\tilde{q}_{3,\overline{m}-1}(\psi_1, \chi_1, \xi_2)) \\
& + (-\tilde{c}^2\bar{\psi}^1 + \tilde{c}^4\bar{\chi}^1)(\bar{\psi}^2\bar{\chi}^1 - \bar{\chi}^2\bar{\psi}^1)\psi_2\tilde{p}_{4,\overline{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\
& - (\tilde{c}^1\bar{\chi}^1 - \tilde{c}^3\bar{\psi}^1)(\bar{\psi}^2\bar{\chi}^1 - \bar{\chi}^2\bar{\psi}^1)\chi_2\tilde{p}_{4,\overline{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) \\
& - (-\tilde{c}^2\psi_1 + \tilde{c}^3\chi_1)(\psi_2\chi_1 - \chi_2\psi_1)\bar{\psi}^2\tilde{q}_{4,\overline{m}-2}(\psi_1, \chi_1, \xi_2) \\
& - (-\tilde{c}^1\chi_1 + \tilde{c}^4\psi_1)(\psi_2\chi_1 - \chi_2\psi_1)\bar{\chi}^2\tilde{q}_{4,\overline{m}-2}(\psi_1, \chi_1, \xi_2) \\
& + \delta_{\overline{m}}^2 s_{\text{gh},1}(\tilde{c}^1\chi_2\bar{\chi}^2 - \tilde{c}^3\chi_2\bar{\psi}^2 + \tilde{c}^2\psi_2\bar{\psi}^2 - \tilde{c}^4\psi_2\bar{\chi}^2)\tilde{b}(\xi_2) \\
& + s_{\text{gh},1}(\eta_{\overline{m}-2} - s_{\text{gh},2}\tilde{\eta}_{\overline{m}-2}).
\end{aligned} \tag{2.70}$$

This equation states that $\omega_{\overline{m}}$ minus the terms in the first six lines on the right hand side is $s_{\text{gh},1}$ -exact. Using the result (2.46) and the explicit form of ω given in (2.64), we infer from equation (2.70):

$$\begin{aligned}
p_{\overline{m}} + q_{\overline{m}} = & (\bar{\psi}^2\bar{\chi}^1 - \bar{\chi}^2\bar{\psi}^1)(\chi_2\tilde{p}_{2,\overline{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2) + \psi_2\tilde{p}_{3,\overline{m}-1}(\bar{\psi}^1, \bar{\chi}^1, \xi_2)) \\
& + (\psi_2\chi_1 - \chi_2\psi_1)(\bar{\chi}^2\tilde{q}_{2,\overline{m}-1}(\psi_1, \chi_1, \xi_2) - \bar{\psi}^2\tilde{q}_{3,\overline{m}-1}(\psi_1, \chi_1, \xi_2)),
\end{aligned} \tag{2.71}$$

$$h_{\overline{m}-1} = (\bar{\psi}^2\bar{\chi}^1 - \bar{\chi}^2\bar{\psi}^1)\tilde{p}_{4,\overline{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2), \tag{2.72}$$

$$g_{\overline{m}-1} = -(\psi_2\chi_1 - \chi_2\psi_1)\tilde{q}_{4,\overline{m}-2}(\psi_1, \chi_1, \xi_2), \tag{2.73}$$

$$\delta_{\overline{m}}^2 b(\xi_2) = 0 \tag{2.74}$$

where $p_{\overline{m}}$ denotes the contribution to $p(\bar{\psi}^1, \bar{\chi}^1, \xi_2)$ with ξ_1 -degree \overline{m} et cetera.

Using the results (2.71) to (2.74) in (2.70), the latter yields furthermore

$$s_{\text{gh},1}(\eta_{\overline{m}-2} - s_{\text{gh},2}\tilde{\eta}_{\overline{m}-2} + \delta_{\overline{m}}^2(\tilde{c}^1\chi_2\bar{\chi}^2 - \tilde{c}^3\chi_2\bar{\psi}^2 + \tilde{c}^2\psi_2\bar{\psi}^2 - \tilde{c}^4\psi_2\bar{\chi}^2)\tilde{b}(\xi_2)) = 0. \tag{2.75}$$

Equation (2.75) implies that $\eta_{\overline{m}-2} - s_{\text{gh},2}\tilde{\eta}_{\overline{m}-2} + \delta_{\overline{m}}^2(\tilde{c}^1\chi_2\bar{\chi}^2 - \tilde{c}^3\chi_2\bar{\psi}^2 + \tilde{c}^2\psi_2\bar{\psi}^2 - \tilde{c}^4\psi_2\bar{\chi}^2)\tilde{b}(\xi_2)$ is of the same form as $\eta_{\overline{m}}$ in equation (2.69). One can continue the analysis of equations (2.67) and (2.68) analogously to lower ξ_1 -degrees. This yields results analogous to (2.71) to (2.74) for the other contributions p_m , q_m , h_m , g_m to the polynomials p , q , h , g in ω and completes the proof of lemma 2.6. \blacksquare

2.1.3 $H_{\text{gh}}(s_{\text{gh}})$ for $N > 2$

We shall determine $H_{\text{gh}}(s_{\text{gh}})$ for $N > 2$ using the results for $N = 2$ by a strategy analogous to the strategy we have used to determine $H_{\text{gh}}(s_{\text{gh}})$ for $N = 2$ by means of the results for $N = 1$ in section 2.1.2 and shall first prove the following result:

Lemma 2.7. *The general solution of the cocycle condition in $H_{\text{gh}}(s_{\text{gh}})$ for $N > 2$ in the spinor representation (1.5) is:*

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim p(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \quad (2.76)$$

with an arbitrary polynomial $p(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N)$ in $\bar{\psi}^1, \bar{\chi}^1, \psi_2, \dots, \psi_N, \chi_2, \dots, \chi_N, \bar{\psi}^2, \dots, \bar{\psi}^N, \bar{\chi}^2, \dots, \bar{\chi}^N$, and an arbitrary polynomial $q(\psi_1, \chi_1, \xi_2, \dots, \xi_N)$ in $\psi_1, \chi_1, \psi_2, \dots, \psi_N, \chi_2, \dots, \chi_N, \bar{\psi}^2, \dots, \bar{\psi}^N, \bar{\chi}^2, \dots, \bar{\chi}^N$.

Proof: We split the coboundary operator s_{gh} according to

$$s_{\text{gh}} = s_{\text{gh}, N=2} + s_{\text{gh}, N>2} \quad (2.77)$$

into a first operator $s_{\text{gh}, N=2}$ which acts like s_{gh} in the case $N = 2$, and a second operator $s_{\text{gh}, N>2}$ which contains the remaining terms of s_{gh} :

$$\begin{aligned} s_{\text{gh}, N=2} &= \sum_{i=1}^2 \left(\psi_i \bar{\psi}^i \frac{\partial}{\partial \bar{c}^1} + \chi_i \bar{\chi}^i \frac{\partial}{\partial \bar{c}^2} + \psi_i \bar{\chi}^i \frac{\partial}{\partial \bar{c}^3} + \chi_i \bar{\psi}^i \frac{\partial}{\partial \bar{c}^4} \right), \\ s_{\text{gh}, N>2} &= \sum_{i=3}^N \left(\psi_i \bar{\psi}^i \frac{\partial}{\partial \bar{c}^1} + \chi_i \bar{\chi}^i \frac{\partial}{\partial \bar{c}^2} + \psi_i \bar{\chi}^i \frac{\partial}{\partial \bar{c}^3} + \chi_i \bar{\psi}^i \frac{\partial}{\partial \bar{c}^4} \right). \end{aligned} \quad (2.78)$$

We denote by $N_{N=2}$ the counting operator which measures the degree of homogeneity in the components of the supersymmetry ghosts ξ_1 and ξ_2 , and by $N_{N>2}$ the counting operator which measures the degree homogeneity in the components of the remaining supersymmetry ghosts ξ_3, \dots, ξ_N :

$$N_{N=2} = \sum_{i=1}^2 \xi_i^\alpha \frac{\partial}{\partial \xi_i^\alpha}, \quad N_{N>2} = \sum_{i=3}^N \xi_i^\alpha \frac{\partial}{\partial \xi_i^\alpha}. \quad (2.79)$$

The first operator $s_{\text{gh}, N=2}$ and the second operator $s_{\text{gh}, N>2}$ and these counting operators fulfill an algebra analogous to (2.52):

$$\begin{aligned} (s_{\text{gh}, N=2})^2 &= \{s_{\text{gh}, N=2}, s_{\text{gh}, N>2}\} = (s_{\text{gh}, N>2})^2 = 0, \\ [N_{N=2}, s_{\text{gh}, N=2}] &= 2s_{\text{gh}, N=2}, \quad [N_{N=2}, s_{\text{gh}, N>2}] = 0, \\ [N_{N>2}, s_{\text{gh}, N=2}] &= 0, \quad [N_{N>2}, s_{\text{gh}, N>2}] = 2s_{\text{gh}, N=2}. \end{aligned} \quad (2.80)$$

The elements $\omega \in \Omega_{\text{gh}}$ are decomposed into eigenfunctions ω_m of $N_{N=2}$ with eigenvalue m ,

$$\omega = \sum_{m=0}^{\bar{m}} \omega_m, \quad N_{N=2} \omega_m = m \omega_m, \quad (2.81)$$

and the cocycle condition $s_{\text{gh}}\omega = 0$ in $H_{\text{gh}}(s_{\text{gh}})$ is decomposed accordingly into

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \begin{cases} s_{\text{gh}, N=2} \omega_{\bar{m}} = 0, & s_{\text{gh}, N=2} \omega_{\bar{m}-2} + s_{\text{gh}, N>2} \omega_{\bar{m}} = 0, & \dots \\ s_{\text{gh}, N=2} \omega_{\bar{m}-1} = 0, & s_{\text{gh}, N=2} \omega_{\bar{m}-3} + s_{\text{gh}, N>2} \omega_{\bar{m}-1} = 0, & \dots \end{cases} \quad (2.82)$$

where, as in equations (2.54), the equations in the first and second line are analysed independently and analogously to each other. Using lemma 2.5 we infer from equation $s_{\text{gh},N=2}\omega_{\overline{m}} = 0$ in (2.82) that

$$\begin{aligned}\omega_{\overline{m}} &= s_{\text{gh},N=2}\eta_{\overline{m}-2} + p_{\overline{m}}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q_{\overline{m}}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \\ &\quad + (-\tilde{c}^2\bar{\psi}^1\psi_2 + \tilde{c}^4\bar{\chi}^1\psi_2 - \tilde{c}^1\bar{\chi}^1\chi_2 + \tilde{c}^3\bar{\psi}^1\chi_2)h_{\overline{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) \\ &\quad + (-\tilde{c}^2\psi_1\bar{\psi}^2 + \tilde{c}^3\chi_1\bar{\psi}^2 - \tilde{c}^1\chi_1\bar{\chi}^2 + \tilde{c}^4\psi_1\bar{\chi}^2)g_{\overline{m}-2}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \\ &\quad + (\tilde{c}^1\chi_1\bar{\chi}^1 - \tilde{c}^3\chi_1\bar{\psi}^1 + \tilde{c}^2\psi_1\bar{\psi}^1 - \tilde{c}^4\psi_1\bar{\chi}^1 \\ &\quad - \tilde{c}^1\chi_2\bar{\chi}^2 + \tilde{c}^3\chi_2\bar{\psi}^2 - \tilde{c}^2\psi_2\bar{\psi}^2 + \tilde{c}^4\psi_2\bar{\chi}^2)b_{\overline{m}-2}(\xi_2, \dots, \xi_N).\end{aligned}\quad (2.83)$$

Using the result (2.83) for $\omega_{\overline{m}}$ in the equation $s_{\text{gh},N=2}\omega_{\overline{m}-2} + s_{\text{gh},N>2}\omega_{\overline{m}} = 0$ in (2.82), we obtain

$$\begin{aligned}0 &= s_{\text{gh},N=2}(\omega_{\overline{m}-2} - s_{\text{gh},N>2}\eta_{\overline{m}-2}) \\ &\quad + h_{\overline{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) \sum_{i=3}^N (-\chi_i\bar{\chi}^i\bar{\psi}^1\psi_2 + \chi_i\bar{\psi}^i\bar{\chi}^1\psi_2 - \psi_i\bar{\psi}^i\bar{\chi}^1\chi_2 \\ &\quad + \psi_i\bar{\chi}^i\bar{\psi}^1\chi_2) + g_{\overline{m}-2}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \sum_{i=3}^N (-\chi_i\bar{\chi}^i\psi_1\bar{\psi}^2 + \psi_i\bar{\chi}^i\chi_1\bar{\psi}^2 \\ &\quad - \psi_i\bar{\psi}^i\chi_1\bar{\chi}^2 + \chi_i\bar{\psi}^i\psi_1\bar{\chi}^2) + b_{\overline{m}-2}(\xi_2, \dots, \xi_N) \sum_{i=3}^N (\psi_i\bar{\psi}^i(\chi_1\bar{\chi}^1 - \chi_2\bar{\chi}^2) \\ &\quad - \psi_i\bar{\chi}^i(\chi_1\bar{\psi}^1 - \chi_2\bar{\psi}^2) + \chi_i\bar{\chi}^i(\psi_1\bar{\psi}^1 - \psi_2\bar{\psi}^2) - \chi_i\bar{\psi}^i(\psi_1\bar{\chi}^1 - \psi_2\bar{\chi}^2)) \\ &= s_{\text{gh},N=2}(\omega_{\overline{m}-2} - s_{\text{gh},N>2}\eta_{\overline{m}-2}) \\ &\quad + h_{\overline{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) \sum_{i=3}^N (\psi_i\chi_2 - \chi_i\psi_2)(\bar{\chi}^i\bar{\psi}^1 - \bar{\psi}^i\bar{\chi}^1) \\ &\quad + g_{\overline{m}-2}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \sum_{i=3}^N (\psi_i\chi_1 - \chi_i\psi_1)(\bar{\chi}^i\bar{\psi}^2 - \bar{\psi}^i\bar{\chi}^2) \\ &\quad + s_{\text{gh},N=2}\left(b_{\overline{m}-2}(\xi_2, \dots, \xi_N) \sum_{i=3}^N (\psi_i\bar{\psi}^i\tilde{c}^2 - \psi_i\bar{\chi}^i\tilde{c}^4 + \chi_i\bar{\chi}^i\tilde{c}^1 - \chi_i\bar{\psi}^i\tilde{c}^3)\right) \\ &\quad - 2b_{\overline{m}-2}(\xi_2, \dots, \xi_N) \sum_{i=3}^N (\chi_i\psi_2 - \psi_i\chi_2)(\bar{\chi}^i\bar{\psi}^2 - \bar{\psi}^i\bar{\chi}^2).\end{aligned}\quad (2.84)$$

Using lemma 2.6 we conclude from equation (2.84):

$$\begin{aligned}&h_{\overline{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) \sum_{i=3}^N (\psi_i\chi_2 - \chi_i\psi_2)(\bar{\chi}^i\bar{\psi}^1 - \bar{\psi}^i\bar{\chi}^1) \\ &+ g_{\overline{m}-2}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \sum_{i=3}^N (\psi_i\chi_1 - \chi_i\psi_1)(\bar{\chi}^i\bar{\psi}^2 - \bar{\psi}^i\bar{\chi}^2)\end{aligned}$$

$$\begin{aligned}
& -2b_{\overline{m}-2}(\xi_2, \dots, \xi_N) \sum_{i=3}^N (\chi_i \psi_2 - \psi_i \chi_2) (\bar{\chi}^i \bar{\psi}^2 - \bar{\psi}^i \bar{\chi}^2) \\
& = (\bar{\psi}^2 \bar{\chi}^1 - \bar{\chi}^2 \bar{\psi}^1) (\chi_2 \tilde{p}_{2, \overline{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + \psi_2 \tilde{p}_{3, \overline{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N)) \\
& \quad + (\psi_2 \chi_1 - \chi_2 \psi_1) (\bar{\chi}^2 \tilde{q}_{2, \overline{m}-2}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) - \bar{\psi}^2 \tilde{q}_{3, \overline{m}-2}(\psi_1, \chi_1, \xi_2, \dots, \xi_N))
\end{aligned} \tag{2.85}$$

for some polynomials $\tilde{p}_{2, \overline{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N), \dots, \tilde{q}_{3, \overline{m}-2}(\psi_1, \chi_1, \xi_2, \dots, \xi_N)$. By inspecting the dependence on $\psi_1, \chi_1, \bar{\psi}^1$ and $\bar{\chi}^1$ of the various terms in equation (2.85), we obtain

$$\begin{aligned}
h_{\overline{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) &= (\bar{\psi}^2 \bar{\chi}^1 - \bar{\chi}^2 \bar{\psi}^1) u_{\overline{m}-4}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N), \\
g_{\overline{m}-2}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) &= (\psi_2 \chi_1 - \chi_2 \psi_1) v_{\overline{m}-4}(\psi_1, \chi_1, \xi_2, \dots, \xi_N), \\
b_{\overline{m}-2}(\xi_2, \dots, \xi_N) &= 0
\end{aligned} \tag{2.86}$$

for some polynomials $u_{\overline{m}-4}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N)$ and $v_{\overline{m}-4}(\psi_1, \chi_1, \xi_2, \dots, \xi_N)$.

Using equations (2.86) in (2.83) yields

$$\begin{aligned}
\omega_{\overline{m}} &= s_{\text{gh}, N=2} \eta_{\overline{m}-2} + p_{\overline{m}}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q_{\overline{m}}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \\
&+ (\tilde{c}^4 \bar{\chi}^1 \psi_2 - \tilde{c}^2 \bar{\psi}^1 \psi_2 + \tilde{c}^3 \bar{\psi}^1 \chi_2 - \tilde{c}^1 \bar{\chi}^1 \chi_2) (\bar{\psi}^2 \bar{\chi}^1 - \bar{\chi}^2 \bar{\psi}^1) u_{\overline{m}-4} \\
&+ (\tilde{c}^3 \chi_1 \bar{\psi}^2 - \tilde{c}^2 \psi_1 \bar{\psi}^2 + \tilde{c}^4 \psi_1 \bar{\chi}^2 - \tilde{c}^1 \chi_1 \bar{\chi}^2) (\psi_2 \chi_1 - \chi_2 \psi_1) v_{\overline{m}-4} \\
&= s_{\text{gh}, N=2} \tilde{\eta}_{\overline{m}-2} + p_{\overline{m}}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q_{\overline{m}}(\psi_1, \chi_1, \xi_2, \dots, \xi_N)
\end{aligned} \tag{2.87}$$

where the arguments of $u_{\overline{m}-4}$ and $v_{\overline{m}-4}$ have been left out and

$$\begin{aligned}
\tilde{\eta}_{\overline{m}-2} &= \eta_{\overline{m}-2} + (\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1) (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) u_{\overline{m}-4} \\
&+ (\tilde{c}^3 \chi_1 - \tilde{c}^2 \psi_1) (\tilde{c}^4 \psi_1 - \tilde{c}^1 \chi_1) v_{\overline{m}-4}.
\end{aligned} \tag{2.88}$$

According to (2.87), $\omega_{\overline{m}}$ is $s_{\text{gh}, N=2}$ -exact up to (possibly) terms $p_{\overline{m}}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q_{\overline{m}}(\psi_1, \chi_1, \xi_2, \dots, \xi_N)$. The latter terms are s_{gh} -closed. Hence, the polynomial

$$\omega' = \omega - s_{\text{gh}} \tilde{\eta}_{\overline{m}-2} - p_{\overline{m}}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) - q_{\overline{m}}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \tag{2.89}$$

is s_{gh} -closed and its decomposition (2.81) contains only terms with $N_{N=2}$ -eigenvalues $m < \overline{m}$. ω' is then treated as ω before, leading to a result analogous to (2.87) for the contribution $\omega'_{\overline{m}}$ with highest $N_{N=2}$ -eigenvalue \overline{m}' contained in ω' (where $\overline{m}' < \overline{m}$). Continuing the arguments, one concludes that ω is s_{gh} -exact except, possibly, for terms $p(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q(\psi_1, \chi_1, \xi_2, \dots, \xi_N) = \sum_{m=0}^{\overline{m}} (p_m(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q_m(\psi_1, \chi_1, \xi_2, \dots, \xi_N))$. This yields lemma 2.7. \blacksquare

To complete the computation of $H_{\text{gh}}(s_{\text{gh}})$ for $N > 2$ we still have to determine those polynomials $p(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q(\psi_1, \chi_1, \xi_2, \dots, \xi_N)$ which are s_{gh} -exact. The solution to this problem is the following lemma 2.8 which together with lemma 2.7 provides an exhaustive characterization of $H_{\text{gh}}(s_{\text{gh}})$ for $N > 2$ in the spinor representations (1.5).

Lemma 2.8 (Coboundaries in lemma 2.7).

$$\begin{aligned}
& p(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \sim 0 \\
& \Leftrightarrow p(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q(\psi_1, \chi_1, \xi_2, \dots, \xi_N) = s_{\text{gh}} \hat{\eta}, \\
& \hat{\eta} = \left[(\tilde{c}^1 \bar{\chi}^1 \chi_1 - \tilde{c}^3 \bar{\psi}^1 \chi_1 + \tilde{c}^2 \bar{\psi}^1 \psi_1 - \tilde{c}^4 \bar{\chi}^1 \psi_1) \right. \\
& \quad - \sum_{i=2}^N (\tilde{c}^1 \bar{\chi}^i \chi_i - \tilde{c}^3 \bar{\psi}^i \chi_i + \tilde{c}^2 \bar{\psi}^i \psi_i - \tilde{c}^4 \bar{\chi}^i \psi_i) \Big] \hat{b}(\xi_2, \dots, \xi_N) \\
& \quad + (\tilde{c}^4 \bar{\chi}^1 - \tilde{c}^2 \bar{\psi}^1) \tilde{p}_2(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) \\
& \quad + (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1) \tilde{p}_3(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) \\
& \quad + (\tilde{c}^3 \chi_1 - \tilde{c}^2 \psi_1) \tilde{q}_2(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \\
& \quad + (\tilde{c}^4 \psi_1 - \tilde{c}^1 \chi_1) \tilde{q}_3(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \\
& \Leftrightarrow p(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q(\psi_1, \chi_1, \xi_2, \dots, \xi_N) = \\
& \quad - \hat{b}(\xi_2, \dots, \xi_N) \sum_{i=2}^N \sum_{j=2}^N (\bar{\psi}^i \bar{\chi}^j - \bar{\chi}^i \bar{\psi}^j) (\psi_i \chi_j - \chi_i \psi_j) \\
& \quad + \sum_{i=2}^N \left[(\psi_i \chi_1 - \chi_i \psi_1) (\bar{\chi}^i \tilde{q}_2(\psi_1, \chi_1, \xi_2, \dots, \xi_N) - \bar{\psi}^i \tilde{q}_3(\psi_1, \chi_1, \xi_2, \dots, \xi_N)) \right. \\
& \quad \left. + (\bar{\psi}^i \bar{\chi}^1 - \bar{\chi}^i \bar{\psi}^1) (\chi_i \tilde{p}_2(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + \psi_i \tilde{p}_3(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N)) \right]. \quad (2.90)
\end{aligned}$$

Proof: We study the coboundary condition

$$p(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q(\psi_1, \chi_1, \xi_2, \dots, \xi_N) = s_{\text{gh}} \eta \quad (2.91)$$

by decomposing it according to $N_{N=2}$ -eigenvalues, using

$$\begin{aligned}
p(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) &= \sum_{m=0}^{\bar{m}} p_m, \quad N_{N=2} p_m = m p_m, \\
q(\psi_1, \chi_1, \xi_2, \dots, \xi_N) &= \sum_{m=0}^{\bar{m}} q_m, \quad N_{N=2} q_m = m q_m, \\
\eta &= \sum_{m=0}^{\bar{p}} \eta_m, \quad N_{N=2} \eta_m = m \eta_m \quad (2.92)
\end{aligned}$$

where $\eta_{\bar{p}}$ does not vanish and $p_{\bar{m}}$ or $q_{\bar{m}}$ do not vanish. By arguments as in the text following equations (2.65) we can assume with no loss of generality that $\bar{m} - 2 \leq \bar{p} \leq \bar{m}$.

In the case $\bar{p} = \bar{m}$, the coboundary condition (2.91) yields at $N_{N=2}$ -eigenvalues $\bar{m} + 2$ and $\bar{m} + 1$

$$s_{\text{gh}, N=2} \eta_{\bar{m}} = 0, \quad s_{\text{gh}, N=2} \eta_{\bar{m}-1} = 0. \quad (2.93)$$

Any $s_{\text{gh},N=2}$ -exact contribution $s_{\text{gh},N=2}\mathcal{Q}_{\bar{p}-2}$ to $\eta_{\bar{p}}$ can be removed from η by replacing η with $\eta - s_{\text{gh}}\mathcal{Q}_{\bar{p}-2}$ as this replacement does not affect the coboundary condition (2.91) (owing to $s_{\text{gh}}^2 = 0$). Therefore, using lemma 2.5 and a notation as above, we infer from the first equation (2.93) that with no loss of generality we can assume

$$\begin{aligned}\eta_{\bar{m}} &= \hat{p}_{\bar{m}}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + \hat{q}_{\bar{m}}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \\ &\quad + (-\tilde{c}^2\bar{\psi}^1\psi_2 + \tilde{c}^4\bar{\chi}^1\psi_2 - \tilde{c}^1\bar{\chi}^1\chi_2 + \tilde{c}^3\bar{\psi}^1\chi_2)\hat{h}_{\bar{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) \\ &\quad + (-\tilde{c}^2\psi_1\bar{\psi}^2 + \tilde{c}^3\chi_1\bar{\psi}^2 - \tilde{c}^1\chi_1\bar{\chi}^2 + \tilde{c}^4\psi_1\bar{\chi}^2)\hat{g}_{\bar{m}-2}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \\ &\quad + (\tilde{c}^1\chi_1\bar{\chi}^1 - \tilde{c}^3\chi_1\bar{\psi}^1 + \tilde{c}^2\psi_1\bar{\psi}^1 - \tilde{c}^4\psi_1\bar{\chi}^1 \\ &\quad - \tilde{c}^1\chi_2\bar{\chi}^2 + \tilde{c}^3\chi_2\bar{\psi}^2 - \tilde{c}^2\psi_2\bar{\psi}^2 + \tilde{c}^4\psi_2\bar{\chi}^2)\hat{b}_{\bar{m}-2}(\xi_2, \dots, \xi_N).\end{aligned}\quad (2.94)$$

The second equation (2.93) implies an analogous result for $\eta_{\bar{m}-1}$.

At $N_{N=2}$ -eigenvalue \bar{m} , the coboundary condition (2.91) yields in the case $\bar{p} = \bar{m}$

$$p_{\bar{m}} + q_{\bar{m}} = s_{\text{gh},N=2}\eta_{\bar{m}-2} + s_{\text{gh},N>2}\eta_{\bar{m}}. \quad (2.95)$$

Using the result (2.94) in equation (2.95), one obtains analogously to equation (2.84)

$$\begin{aligned}p_{\bar{m}} + q_{\bar{m}} &= s_{\text{gh},N=2}\eta_{\bar{m}-2} \\ &\quad + \hat{h}_{\bar{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) \sum_{i=3}^N (\bar{\psi}^i\bar{\chi}^1 - \bar{\chi}^i\bar{\psi}^1)(\chi_i\psi_2 - \psi_i\chi_2) \\ &\quad + \hat{g}_{\bar{m}-2}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \sum_{i=3}^N (\bar{\psi}^i\bar{\chi}^2 - \bar{\chi}^i\bar{\psi}^2)(\chi_i\psi_1 - \psi_i\chi_1) \\ &\quad + s_{\text{gh},N=2} \sum_{i=3}^N \hat{b}_{\bar{m}-2}(\xi_2, \dots, \xi_N) (\tilde{c}^1\chi_i\bar{\chi}^i - \tilde{c}^3\chi_i\bar{\psi}^i + \tilde{c}^2\psi_i\bar{\psi}^i - \tilde{c}^4\psi_i\bar{\chi}^i) \\ &\quad - 2\hat{b}_{\bar{m}-2}(\xi_2, \dots, \xi_N) \sum_{i=3}^N (\bar{\psi}^i\bar{\chi}^2 - \bar{\chi}^i\bar{\psi}^2)(\psi_i\chi_2 - \chi_i\psi_2).\end{aligned}\quad (2.96)$$

We write this equation as

$$p'_{\bar{m}}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q'_{\bar{m}}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) + b'_{\bar{m}}(\xi_2, \dots, \xi_N) = s_{\text{gh},N=2}\eta'_{\bar{m}-2} \quad (2.97)$$

where, leaving out the arguments of $p'_{\bar{m}}$, $q'_{\bar{m}}$ and $b'_{\bar{m}}$,

$$\begin{aligned}p'_{\bar{m}} &= p_{\bar{m}} - \hat{h}_{\bar{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) \sum_{i=3}^N (\bar{\psi}^i\bar{\chi}^1 - \bar{\chi}^i\bar{\psi}^1)(\chi_i\psi_2 - \psi_i\chi_2) \\ q'_{\bar{m}} &= q_{\bar{m}} - \hat{g}_{\bar{m}-2}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \sum_{i=3}^N (\bar{\psi}^i\bar{\chi}^2 - \bar{\chi}^i\bar{\psi}^2)(\chi_i\psi_1 - \psi_i\chi_1) \\ b'_{\bar{m}} &= 2\hat{b}_{\bar{m}-2}(\xi_2, \dots, \xi_N) \sum_{i=3}^N (\bar{\psi}^i\bar{\chi}^2 - \bar{\chi}^i\bar{\psi}^2)(\psi_i\chi_2 - \chi_i\psi_2)\end{aligned}$$

$$\eta'_{\overline{m}-2} = \eta_{\overline{m}-2} + \sum_{i=3}^N \hat{b}_{\overline{m}-2}(\xi_2, \dots, \xi_N) (\tilde{c}^1 \chi_i \bar{\chi}^i - \tilde{c}^3 \chi_i \bar{\psi}^i + \tilde{c}^2 \psi_i \bar{\psi}^i - \tilde{c}^4 \psi_i \bar{\chi}^i). \quad (2.98)$$

According to equation (2.97), $p'_{\overline{m}} + q'_{\overline{m}} + b'_{\overline{m}}$ is a sum of polynomials in the supersymmetry ghosts which either do not depend on ψ_1 and χ_1 or on $\bar{\psi}^1$ and $\bar{\chi}^1$. Using lemma 2.6, we conclude from equation (2.97)

$$\begin{aligned} & p'_{\overline{m}}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q'_{\overline{m}}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) + b'_{\overline{m}}(\xi_2, \dots, \xi_N) \\ &= (\bar{\psi}^2 \bar{\chi}^1 - \bar{\chi}^2 \bar{\psi}^1)(\chi_2 p'_{2, \overline{m}-3}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + \psi_2 p'_{3, \overline{m}-3}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N)) \\ &+ (\psi_2 \chi_1 - \chi_2 \psi_1)(\bar{\chi}^2 q'_{2, \overline{m}-3}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) - \bar{\psi}^2 q'_{3, \overline{m}-3}(\psi_1, \chi_1, \xi_2, \dots, \xi_N)). \end{aligned} \quad (2.99)$$

Equations (2.98) and (2.99) yield

$$\begin{aligned} p_{\overline{m}} + q_{\overline{m}} &= -2\hat{b}_{\overline{m}-2}(\xi_2, \dots, \xi_N) \sum_{i=3}^N (\bar{\psi}^i \bar{\chi}^2 - \bar{\chi}^i \bar{\psi}^2)(\psi_i \chi_2 - \chi_i \psi_2) \\ &+ (\bar{\psi}^2 \bar{\chi}^1 - \bar{\chi}^2 \bar{\psi}^1)(\chi_2 p'_{2, \overline{m}-3}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + \psi_2 p'_{3, \overline{m}-3}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N)) \\ &+ (\psi_2 \chi_1 - \chi_2 \psi_1)(\bar{\chi}^2 q'_{2, \overline{m}-3}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) - \bar{\psi}^2 q'_{3, \overline{m}-3}(\psi_1, \chi_1, \xi_2, \dots, \xi_N)) \\ &+ \hat{h}_{\overline{m}-2}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) \sum_{i=3}^N (\bar{\psi}^i \bar{\chi}^1 - \bar{\chi}^i \bar{\psi}^1)(\chi_i \psi_2 - \psi_i \chi_2) \\ &+ \hat{g}_{\overline{m}-2}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) \sum_{i=3}^N (\bar{\psi}^i \bar{\chi}^2 - \bar{\chi}^i \bar{\psi}^2)(\chi_i \psi_1 - \psi_i \chi_1) \\ &= s_{\text{gh}} \tilde{\eta} + \hat{b}_{\overline{m}-2}(\xi_2, \dots, \xi_N) \sum_{i=3}^N \sum_{j=3}^N (\bar{\psi}^i \bar{\chi}^j - \bar{\chi}^i \bar{\psi}^j)(\psi_i \chi_j - \chi_i \psi_j) \\ &- \sum_{i=3}^N \left[(\psi_i \chi_1 - \chi_i \psi_1)(\bar{\chi}^i q'_{2, \overline{m}-3}(\psi_1, \chi_1, \xi_2, \dots, \xi_N) - \bar{\psi}^i q'_{3, \overline{m}-3}(\psi_1, \chi_1, \xi_2, \dots, \xi_N)) \right. \\ &\left. + (\bar{\psi}^i \bar{\chi}^1 - \bar{\chi}^i \bar{\psi}^1)(\chi_i p'_{2, \overline{m}-3}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + \psi_i p'_{3, \overline{m}-3}(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N)) \right] \end{aligned} \quad (2.100)$$

where, leaving out the arguments of $\hat{b}_{\overline{m}-2}$ etc.,

$$\begin{aligned} \tilde{\eta} &= \left[(\tilde{c}^1 \bar{\chi}^1 \chi_1 - \tilde{c}^3 \bar{\psi}^1 \chi_1 + \tilde{c}^2 \bar{\psi}^1 \psi_1 - \tilde{c}^4 \bar{\chi}^1 \psi_1) \right. \\ &- \sum_{i=2}^N (\tilde{c}^1 \bar{\chi}^i \chi_i - \tilde{c}^3 \bar{\psi}^i \chi_i + \tilde{c}^2 \bar{\psi}^i \psi_i - \tilde{c}^4 \bar{\chi}^i \psi_i) \left. \right] \hat{b}_{\overline{m}-2} \\ &+ (\tilde{c}^4 \bar{\chi}^1 - \tilde{c}^2 \bar{\psi}^1)(p'_{2, \overline{m}-3} + \psi_2 \hat{h}_{\overline{m}-2}) + (\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1)(p'_{3, \overline{m}-3} - \chi_2 \hat{h}_{\overline{m}-2}) \\ &+ (\tilde{c}^3 \chi_1 - \tilde{c}^2 \psi_1)(q'_{2, \overline{m}-3} + \bar{\psi}^2 \hat{g}_{\overline{m}-2}) + (\tilde{c}^4 \psi_1 - \tilde{c}^1 \chi_1)(q'_{3, \overline{m}-3} + \bar{\chi}^2 \hat{g}_{\overline{m}-2}). \end{aligned} \quad (2.101)$$

$\tilde{\eta}$ contributes to $\hat{\eta}$ in (2.90) at $N_{N=2}$ -degrees \overline{m} and $\overline{m} - 2$, with $p'_{2, \overline{m}-3} + \psi_2 \hat{h}_{\overline{m}-2}$ contributing to \tilde{p}_2 et cetera. If $\overline{p} = \overline{m} - 2$ or $\overline{p} = \overline{m} - 1$, the coboundary condition

(2.91) yields $p_{\overline{m}} + q_{\overline{m}} = s_{\text{gh}, N=2} \eta_{\overline{m}-2}$ and, in the case $\overline{p} = \overline{m} - 1$, additionally $s_{\text{gh}, N=2} \eta_{\overline{m}-1} = 0$. The latter are equations as (2.93) and (2.95) with $\eta_{\overline{m}} = 0$ and lead to results for $p_{\overline{m}} + q_{\overline{m}}$ as in equations (2.100) and (2.101) with $\hat{h}_{\overline{m}-2} = \hat{g}_{\overline{m}-2} = \hat{b}_{\overline{m}-2} = 0$.

Equation (2.100) shows that $p(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q(\psi_1, \chi_1, \xi_2, \dots, \xi_N) - s_{\text{gh}} \eta'$ is an s_{gh} -exact polynomial in the supersymmetry ghosts of the form $p'(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q'(\psi_1, \chi_1, \xi_2, \dots, \xi_N)$ which contains no terms with $N_{N=2}$ -eigenvalues $m \geq \overline{m}$. Hence, it can be treated as $p(\bar{\psi}^1, \bar{\chi}^1, \xi_2, \dots, \xi_N) + q(\psi_1, \chi_1, \xi_2, \dots, \xi_N)$ above and the process can be continued until the $N_{N=2}$ -eigenvalue drops to zero which yields lemma 2.8. ■

2.2 $H_{\text{gh}}(s_{\text{gh}})$ in covariant form

We shall now provide $\mathfrak{so}(t, 4-t)$ -covariant versions of the results for $D = 4$ which extend these results to all spinor representations equivalent to the particular representations (1.5). To this end we introduce the following $\mathfrak{so}(t, 4-t)$ -covariant ghost polynomials (with $\xi_i^\pm = \frac{1}{2} \xi_i (\mathbb{1} \pm \hat{\Gamma})$):

$$\begin{aligned} \vartheta_i^\alpha &= c^a \xi_i^\beta \Gamma_{a\beta}^\alpha, \\ \vartheta_i^{+\alpha} &= \frac{1}{2} \vartheta_i^\beta (\mathbb{1} + \hat{\Gamma})_{\beta}^\alpha = c^a \xi_i^{-\beta} \Gamma_{a\beta}^\alpha, \\ \vartheta_i^{-\alpha} &= \frac{1}{2} \vartheta_i^\beta (\mathbb{1} - \hat{\Gamma})_{\beta}^\alpha = c^a \xi_i^{+\beta} \Gamma_{a\beta}^\alpha, \\ \Theta_{ij} &= \vartheta_i^+ \cdot \xi_j^+ = \frac{1}{4} \vartheta_i^\gamma (\mathbb{1} + \hat{\Gamma})_{\gamma}^\alpha \xi_j^\delta (\mathbb{1} + \hat{\Gamma})_{\delta}^\beta C_{\alpha\beta}^{-1} = c^a \xi_i^\alpha (\frac{1}{2} \Gamma_a (\mathbb{1} + \hat{\Gamma}) C^{-1})_{\alpha\beta} \xi_j^\beta \end{aligned} \quad (2.102)$$

where $\vartheta_i^+ \cdot \xi_j^+$ denotes the $\mathfrak{so}(t, 4-t)$ -invariant product $\vartheta_i^{+\alpha} C_{\alpha\beta}^{-1} \xi_j^{+\beta}$ of ϑ_i^+ and ξ_j^+ (cf. section 2.6 of [1]). We note that the products $\vartheta_i^- \cdot \xi_j^-$ can be expressed in terms of the products $\vartheta_i^+ \cdot \xi_j^+$ (and vice versa):

$$\begin{aligned} \vartheta_i^+ \cdot \xi_j^+ &= c^a \xi_i^\alpha (\frac{1}{2} \Gamma_a (\mathbb{1} + \hat{\Gamma}) C^{-1})_{\alpha\beta} \xi_j^\beta \\ &= c^a \xi_i^\alpha (\frac{1}{2} \Gamma_a (\mathbb{1} - \hat{\Gamma}) C^{-1})_{\beta\alpha} \xi_j^\beta = \vartheta_j^- \cdot \xi_i^-. \end{aligned} \quad (2.103)$$

The coboundary operator s_{gh} acts on the ϑ_i^\pm and Θ_{ij} according to

$$\begin{aligned} s_{\text{gh}} \vartheta_i^{+\alpha} &= 2i \sum_{j=1}^N (\xi_i^- \cdot \xi_j^-) \xi_j^{+\alpha}, \\ s_{\text{gh}} \vartheta_i^{-\alpha} &= 2i \sum_{j=1}^N (\xi_i^+ \cdot \xi_j^+) \xi_j^{-\alpha}, \\ s_{\text{gh}} \Theta_{ij} &= -2i \sum_{k=1}^N (\xi_i^- \cdot \xi_k^-) (\xi_j^+ \cdot \xi_k^+). \end{aligned} \quad (2.104)$$

In the spinor representations (1.5) one has:

$$\begin{aligned}
(\vartheta_i^1, \vartheta_i^2, \vartheta_i^3, \vartheta_i^4) &= 2(\tilde{c}^1 \bar{\chi}^i - \tilde{c}^3 \bar{\psi}^i, -\tilde{c}^1 \chi_i + \tilde{c}^4 \psi_i, -\tilde{c}^2 \psi_i + \tilde{c}^3 \chi_i, -\tilde{c}^2 \bar{\psi}^i + \tilde{c}^4 \bar{\chi}^i), \\
(\vartheta_i^{+1}, \vartheta_i^{+2}, \vartheta_i^{+3}, \vartheta_i^{+4}) &= (\vartheta_i^1, 0, 0, \vartheta_i^4), \quad (\vartheta_i^{-1}, \vartheta_i^{-2}, \vartheta_i^{-3}, \vartheta_i^{-4}) = (0, \vartheta_i^2, \vartheta_i^3, 0), \\
\Theta_{ij} &= 2i(-\tilde{c}^1 \bar{\chi}^i \chi_j + \tilde{c}^3 \bar{\psi}^i \chi_j - \tilde{c}^2 \bar{\psi}^i \psi_j + \tilde{c}^4 \bar{\chi}^i \psi_j), \\
\xi_i^+ \cdot \xi_j^+ &= -i\psi_i \chi_j + i\chi_i \psi_j, \quad \xi_i^- \cdot \xi_j^- = i\bar{\psi}^i \bar{\chi}^j - i\bar{\chi}^i \bar{\psi}^j, \\
\vartheta_1^+ \cdot \vartheta_1^+ &= 8i(\tilde{c}^1 \bar{\chi}^1 - \tilde{c}^3 \bar{\psi}^1)(\tilde{c}^2 \bar{\psi}^1 - \tilde{c}^4 \bar{\chi}^1), \\
\vartheta_1^- \cdot \vartheta_1^- &= 8i(\tilde{c}^1 \chi_1 - \tilde{c}^4 \psi_1)(\tilde{c}^3 \chi_1 - \tilde{c}^2 \psi_1).
\end{aligned} \tag{2.105}$$

Using these expressions one straightforwardly verifies equations (2.104) in the spinor representations (1.5) which implies that they also hold in any spinor representation equivalent to (1.5) owing to their $\mathfrak{so}(t, 4-t)$ -covariance. Furthermore these expressions show that various ghost polynomials in lemmas 2.4 to 2.8 can be expressed in an $\mathfrak{so}(t, 4-t)$ -covariant way. Using additionally that equivalence transformations relating equivalent spinor representations do not mix chiralities of spinors in the sense of section 2.7 of [1], one can directly obtain from lemmas 2.4 to 2.8 the following results that are valid for all spinor representations equivalent to (1.5).

The covariant version of lemma 2.4 is:

Lemma 2.9 ($H_{\text{gh}}(s_{\text{gh}})$ for $N = 1$).

In the case $N = 1$

(i) any cocycle in $H_{\text{gh}}(s_{\text{gh}})$ is equivalent to a linear combination of a polynomial in the components of ξ_1^- and ϑ_1^+ , of a polynomial in the components of ξ_1^+ and ϑ_1^- , and of Θ_{11} , with ϑ_1^\pm and Θ_{11} as in equations (2.102):

$$\begin{aligned}
s_{\text{gh}}\omega = 0 \Leftrightarrow \omega &\sim \Theta_{11}b + p(\xi_1^-) + q(\xi_1^+) + \vartheta_1^{+\alpha}p_{\underline{\alpha}}(\xi_1^-) + \vartheta_1^{-\alpha}q_{\underline{\alpha}}(\xi_1^+) \\
&\quad + (\vartheta_1^+ \cdot \vartheta_1^+)p_{-}(\xi_1^-) + (\vartheta_1^- \cdot \vartheta_1^-)q_{+}(\xi_1^+)
\end{aligned} \tag{2.106}$$

with arbitrary polynomials $p(\xi_1^-)$, $p_{\underline{\alpha}}(\xi_1^-)$, $p_{-}(\xi_1^-)$ in the components of ξ_1^- , arbitrary polynomials $q(\xi_1^+)$, $q_{\underline{\alpha}}(\xi_1^+)$, $q_{+}(\xi_1^+)$ in the components of ξ_1^+ , and an arbitrary complex number $b \in \mathbb{C}$;

(ii) a linear combination of a polynomial in the components of ξ_1^- and ϑ_1^+ , of a polynomial in the components of ξ_1^+ and ϑ_1^- , and of Θ_{11} is exact in $H_{\text{gh}}(s_{\text{gh}})$ if and only if it vanishes:

$$\begin{aligned}
0 &\sim \Theta_{11}b + p(\xi_1^-) + q(\xi_1^+) + \vartheta_1^{+\alpha}p_{\underline{\alpha}}(\xi_1^-) + \vartheta_1^{-\alpha}q_{\underline{\alpha}}(\xi_1^+) \\
&\quad + (\vartheta_1^+ \cdot \vartheta_1^+)p_{-}(\xi_1^-) + (\vartheta_1^- \cdot \vartheta_1^-)q_{+}(\xi_1^+) \\
\Leftrightarrow \quad b &= p(\xi_1^-) + q(\xi_1^+) = \vartheta_1^{+\alpha}p_{\underline{\alpha}}(\xi_1^-) = \vartheta_1^{-\alpha}q_{\underline{\alpha}}(\xi_1^+) = p_{-}(\xi_1^-) = q_{+}(\xi_1^+) = 0.
\end{aligned} \tag{2.107}$$

Comments:

1. Lemma 2.9 reproduces for signatures (1, 3) and (3, 1) the results derived in section 13.1 of [4] and in [5] when particularized for the spinor representations considered there.

2. In the case $N = 1$ equations (2.104) yield $s_{\text{gh}}\vartheta_1^{+\alpha} = 2i(\xi_1^- \cdot \xi_1^-)\xi_1^{+\alpha} = 0$ (owing to $\xi_i^- \cdot \xi_j^- = -\xi_j^- \cdot \xi_i^-$ which implies $\xi_1^- \cdot \xi_1^- = 0$) and analogously $s_{\text{gh}}\vartheta_1^{-\alpha} = 0$ as well as $s_{\text{gh}}\Theta_{11} = -2i(\xi_1^- \cdot \xi_1^-)(\xi_1^+ \cdot \xi_1^+) = 0$ which shows that $\vartheta_1^{+\alpha}$, $\vartheta_1^{-\alpha}$ and Θ_{11} are indeed cocycles in $H_{\text{gh}}(s_{\text{gh}})$ in the case $N = 1$.

3. The cocycles $p(\xi_1^-)$, $\vartheta_1^{+\alpha}p_{\alpha}(\xi_1^-)$ and $(\vartheta_1^+ \cdot \vartheta_1^+)p_{-}(\xi_1^-)$ depend on the supersymmetry ghosts only via components $\xi_1^{-\alpha}$ of negative chirality since ϑ_1^+ also depends on the supersymmetry ghosts only via the $\xi_1^{-\alpha}$. Analogously the cocycles $q(\xi_1^+)$, $\vartheta_1^{-\alpha}q_{\alpha}(\xi_1^+)$ and $(\vartheta_1^- \cdot \vartheta_1^-)q_{+}(\xi_1^+)$ depend on the supersymmetry ghosts only via components $\xi_1^{+\alpha}$ of positive chirality. In contrast, Θ_{11} depends linearly both on components $\xi_1^{-\alpha}$ with negative chirality and on components $\xi_1^{+\alpha}$ with positive chirality.

Lemmas 2.5 and 2.6 yield:

Lemma 2.10 ($H_{\text{gh}}(s_{\text{gh}})$ for $N = 2$).

In the case $N = 2$

(i) *the general solution of the cocycle condition in $H_{\text{gh}}(s_{\text{gh}})$ is*

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim P(\xi_1^-, \xi_2) + Q(\xi_1^+, \xi_2) + \Theta_{12}H(\xi_1^-, \xi_2) + \Theta_{21}G(\xi_1^+, \xi_2) + (\Theta_{11} - \Theta_{22})B(\xi_2) \quad (2.108)$$

with Θ_{ij} as in equations (2.102), arbitrary polynomials $P(\xi_1^-, \xi_2)$, $H(\xi_1^-, \xi_2)$ in the components of ξ_1^- and ξ_2 , arbitrary polynomials $Q(\xi_1^+, \xi_2)$, $G(\xi_1^+, \xi_2)$ in the components of ξ_1^+ and ξ_2 , and an arbitrary polynomial $B(\xi_2)$ in the components of ξ_2 ;

(ii) *a cocycle $P(\xi_1^-, \xi_2) + Q(\xi_1^+, \xi_2) + \Theta_{12}H(\xi_1^-, \xi_2) + \Theta_{21}G(\xi_1^+, \xi_2) + (\Theta_{11} - \Theta_{22})B(\xi_2)$ is s_{gh} -exact if and only if $P(\xi_1^-, \xi_2) + Q(\xi_1^+, \xi_2)$ is a linear combination of polynomials $(\xi_1^- \cdot \xi_2^-)\xi_2^{+\alpha}\tilde{P}_{\alpha}(\xi_1^-, \xi_2)$ and $(\xi_1^+ \cdot \xi_2^+)\xi_2^{-\alpha}\tilde{Q}_{\alpha}(\xi_1^+, \xi_2)$ and $H(\xi_1^-, \xi_2)$ and $G(\xi_1^+, \xi_2)$ factorize according to $(\xi_1^- \cdot \xi_2^-)\tilde{P}(\xi_1^-, \xi_2)$ and $(\xi_1^+ \cdot \xi_2^+)\tilde{Q}(\xi_1^+, \xi_2)$, respectively, and $B(\xi_2)$ vanishes:*

$$\begin{aligned} 0 &\sim P(\xi_1^-, \xi_2) + Q(\xi_1^+, \xi_2) + \Theta_{12}H(\xi_1^-, \xi_2) + \Theta_{21}G(\xi_1^+, \xi_2) + (\Theta_{11} - \Theta_{22})B(\xi_2) \\ &\Leftrightarrow P(\xi_1^-, \xi_2) + Q(\xi_1^+, \xi_2) = (\xi_1^- \cdot \xi_2^-)\xi_2^{+\alpha}\tilde{P}_{\alpha}(\xi_1^-, \xi_2) + (\xi_1^+ \cdot \xi_2^+)\xi_2^{-\alpha}\tilde{Q}_{\alpha}(\xi_1^+, \xi_2) \wedge \\ &\quad H(\xi_1^-, \xi_2) = (\xi_1^- \cdot \xi_2^-)\tilde{P}(\xi_1^-, \xi_2) \wedge \\ &\quad G(\xi_1^+, \xi_2) = (\xi_1^+ \cdot \xi_2^+)\tilde{Q}(\xi_1^+, \xi_2) \wedge \\ &\quad B(\xi_2) = 0. \end{aligned} \quad (2.109)$$

Comments:

4. The third equation (2.104) yields

$$\begin{aligned} N = 2 : \quad s_{\text{gh}}\Theta_{12} &= 0 = s_{\text{gh}}\Theta_{21}, \\ s_{\text{gh}}\Theta_{11} &= -2i(\xi_1^- \cdot \xi_2^-)(\xi_1^+ \cdot \xi_2^+) = s_{\text{gh}}\Theta_{22}. \end{aligned} \quad (2.110)$$

This verifies that the terms in (2.108) are indeed s_{gh} -closed in the case $N = 2$.

5. The first and second equation (2.104) and equations (2.102) and (2.103) yield

$$N = 2 : \quad (\xi_1^- \cdot \xi_2^-)\xi_2^{+\alpha} = -\frac{i}{2} s_{\text{gh}}\vartheta_1^{+\alpha},$$

$$\begin{aligned}
(\xi_1^+ \cdot \xi_2^+) \xi_2^{-\alpha} &= -\frac{i}{2} s_{\text{gh}} \vartheta_1^{-\alpha}, \\
(\xi_1^- \cdot \xi_2^-) \Theta_{12} &= (\xi_1^- \cdot \xi_2^-) (\vartheta_1^+ \cdot \xi_2^+) = -\frac{i}{2} (\vartheta_1^+ \cdot s_{\text{gh}} \vartheta_1^+) = \frac{i}{4} s_{\text{gh}} (\vartheta_1^+ \cdot \vartheta_1^+), \\
(\xi_1^+ \cdot \xi_2^+) \Theta_{21} &= (\xi_1^+ \cdot \xi_2^+) (\vartheta_1^- \cdot \xi_2^-) = -\frac{i}{2} (\vartheta_1^- \cdot s_{\text{gh}} \vartheta_1^-) = \frac{i}{4} s_{\text{gh}} (\vartheta_1^- \cdot \vartheta_1^-).
\end{aligned} \tag{2.111}$$

These relations underlie the result (2.109).

Lemmas 2.7 and 2.8 yield:

Lemma 2.11 ($H_{\text{gh}}(s_{\text{gh}})$ for $N > 2$).

In the cases $N > 2$

(i) the general solution of the cocycle condition in $H_{\text{gh}}(s_{\text{gh}})$ is

$$s_{\text{gh}} \omega = 0 \Leftrightarrow \omega \sim P(\xi_1^-, \xi_2, \dots, \xi_N) + Q(\xi_1^+, \xi_2, \dots, \xi_N) \tag{2.112}$$

with an arbitrary polynomial $P(\xi_1^-, \xi_2, \dots, \xi_N)$ in the components of $\xi_1^-, \xi_2, \dots, \xi_N$ and an arbitrary polynomial $Q(\xi_1^+, \xi_2, \dots, \xi_N)$ in the components of $\xi_1^+, \xi_2, \dots, \xi_N$;

(ii) a cocycle $P(\xi_1^-, \xi_2, \dots, \xi_N) + Q(\xi_1^+, \xi_2, \dots, \xi_N)$ is s_{gh} -exact if and only if it is the s_{gh} -transformation of a linear combination of $\Theta_{11} - \sum_{i=2}^N \Theta_{ii}$ times a polynomial in the components of ξ_2, \dots, ξ_N , of the components of ϑ_1^+ times polynomials in the components of $\xi_1^-, \xi_2, \dots, \xi_N$, and of the components of ϑ_1^- times polynomials in the components of $\xi_1^+, \xi_2, \dots, \xi_N$:

$$\begin{aligned}
&P(\xi_1^-, \xi_2, \dots, \xi_N) + Q(\xi_1^+, \xi_2, \dots, \xi_N) \sim 0 \\
&\Leftrightarrow P(\xi_1^-, \xi_2, \dots, \xi_N) + Q(\xi_1^+, \xi_2, \dots, \xi_N) \\
&= s_{\text{gh}} \left[(\Theta_{11} - \sum_{i=2}^N \Theta_{ii}) \hat{B}(\xi_2, \dots, \xi_N) + \vartheta_1^{+\alpha} \tilde{P}_{\alpha}(\xi_1^-, \xi_2, \dots, \xi_N) \right. \\
&\quad \left. + \vartheta_1^{-\alpha} \tilde{Q}_{\alpha}(\xi_1^+, \xi_2, \dots, \xi_N) \right] \tag{2.113}
\end{aligned}$$

$$\begin{aligned}
&= 2i \sum_{i=2}^N \sum_{j=2}^N (\xi_i^+ \cdot \xi_j^+) (\xi_i^- \cdot \xi_j^-) \hat{B}(\xi_2, \dots, \xi_N) \\
&+ 2i \sum_{i=2}^N \left[(\xi_1^- \cdot \xi_i^-) \xi_i^{+\alpha} \tilde{P}_{\alpha}(\xi_1^-, \xi_2, \dots, \xi_N) + (\xi_1^+ \cdot \xi_i^+) \xi_i^{-\alpha} \tilde{Q}_{\alpha}(\xi_1^+, \xi_2, \dots, \xi_N) \right].
\end{aligned} \tag{2.114}$$

Comment:

6. One has

$$s_{\text{gh}}(\Theta_{11} - \sum_{i=2}^N \Theta_{ii}) = 2i \sum_{i=2}^N \sum_{j=2}^N (\xi_i^+ \cdot \xi_j^+) (\xi_i^- \cdot \xi_j^-).$$

This shows that for $N > 2$ there are polynomials in the supersymmetry ghosts which do not depend on components of ξ_1 and are nevertheless s_{gh} -exact. According to part (ii) of lemma 2.11, these polynomials are of the form $s_{\text{gh}}(\Theta_{11} - \sum_{i=2}^N \Theta_{ii}) \hat{B}(\xi_2, \dots, \xi_N)$.

3 Primitive elements in five dimensions

3.1 Relation of s_{gh} -transformations in $D = 4$ and $D = 5$

We shall use the results in $D = 4$ dimensions to derive the results in $D = 5$ dimensions. To this end we first relate the s_{gh} -transformations in $D = 4$ and $D = 5$ by marking $D = 4$ objects by a subscript ($D = 4$). For spinor representations in $D = 4$ and $D = 5$ with the same gamma-matrices $\Gamma^1, \dots, \Gamma^4$, the $D = 5$ charge conjugation matrix C and Γ_5 are related to the $D = 4$ charge conjugation matrix $C_{(D=4)}$ and $\hat{\Gamma}_{(D=4)}$, respectively, by:

$$C = C_{(D=4)} \hat{\Gamma}_{(D=4)}, \quad \Gamma_5 = k_5 \hat{\Gamma}_{(D=4)}. \quad (3.1)$$

Decomposing the $D = 5$ supersymmetry ghosts according to

$$\xi_i = \xi_i^+ + \xi_i^-, \quad \xi_i^\pm = \frac{1}{2} \xi_i (\mathbb{1} \pm \hat{\Gamma}_{(D=4)}), \quad (3.2)$$

we have in $D = 5$, using matrix notation with $\xi_i = (\xi_i^1, \xi_i^2, \xi_i^3, \xi_i^4)$:

$$\begin{aligned} a \in \{1, \dots, 4\} : s_{\text{gh}} c^a &= i \sum_{k=1}^{N/2} \xi_{2k-1} \Gamma^a C^{-1} (\xi_{2k})^\top \\ &= i \sum_{k=1}^{N/2} \left(\xi_{2k-1}^- \Gamma^a C_{(D=4)}^{-1} (\xi_{2k}^+)^{\top} - \xi_{2k-1}^+ \Gamma^a C_{(D=4)}^{-1} (\xi_{2k}^-)^{\top} \right). \end{aligned} \quad (3.3)$$

In $D = 4$ we have:

$$s_{\text{gh}} c_{(D=4)}^a = \frac{i}{2} \sum_{i=1}^N \xi_{i(D=4)} \Gamma^a C_{(D=4)}^{-1} (\xi_{i(D=4)})^\top = i \sum_{i=1}^N \xi_{i(D=4)}^+ \Gamma^a C_{(D=4)}^{-1} (\xi_{i(D=4)}^-)^{\top}. \quad (3.4)$$

Comparing (3.3) and (3.4) we observe that we can match the s_{gh} -transformations of c^1, \dots, c^4 in $D = 4$ and $D = 5$ for $N \in \{2, 4, 6, \dots\}$ using the identifications

$$\begin{aligned} a = 1, \dots, 4 : c^a &= c_{(D=4)}^a; \\ k = 1, \dots, \frac{N}{2} : \xi_{2k-1}^+ &= \xi_{2k-1(D=4)}^+, \quad \xi_{2k-1}^- = \xi_{2k(D=4)}^-, \quad \xi_{2k-1} = \xi_{2k-1(D=4)}^+ + \xi_{2k(D=4)}^-, \\ \xi_{2k}^+ &= \xi_{2k(D=4)}^+, \quad \xi_{2k}^- = -\xi_{2k-1(D=4)}^-, \quad \xi_{2k} = \xi_{2k(D=4)}^+ - \xi_{2k-1(D=4)}^-. \end{aligned} \quad (3.5)$$

With these identifications we obtain

$$\begin{aligned} s_{\text{gh}} c^5 &= i \sum_{k=1}^{N/2} \xi_{2k-1} \Gamma^5 C^{-1} (\xi_{2k})^\top = \frac{i}{k_5} \sum_{k=1}^{N/2} \xi_{2k-1} C_{(D=4)}^{-1} (\xi_{2k})^\top \\ &= \frac{i}{k_5} \sum_{k=1}^{N/2} \left(\xi_{2k-1}^+ C_{(D=4)}^{-1} (\xi_{2k}^+)^{\top} + \xi_{2k-1}^- C_{(D=4)}^{-1} (\xi_{2k}^-)^{\top} \right) \end{aligned}$$

$$\begin{aligned}
&= \frac{i}{k_5} \sum_{k=1}^{N/2} \left[\xi_{2k-1}^+ C^{-1} (\xi_{2k}^+)^{\top} - \xi_{2k}^- C^{-1} (\xi_{2k-1}^-)^{\top} \right]_{(D=4)} \\
&= \frac{i}{k_5} \sum_{k=1}^{N/2} [\xi_{2k-1}^+ \cdot \xi_{2k}^+ + \xi_{2k-1}^- \cdot \xi_{2k}^-]_{(D=4)} = \frac{i}{k_5} \sum_{k=1}^{N/2} [\xi_{2k-1} \cdot \xi_{2k}]_{(D=4)} \quad (3.6)
\end{aligned}$$

with all terms within parantheses $[\dots]_{(D=4)}$ referring to $D = 4$. In particular, spinor products within such parantheses, such as $[\xi_{2k-1} \cdot \xi_{2k}]_{(D=4)}$, refer to $D = 4$ spinor products of $D = 4$ spinors, i.e.

$$[\xi_{2k-1} \cdot \xi_{2k}]_{(D=4)} = \xi_{2k-1(D=4)} C_{(D=4)}^{-1} (\xi_{2k(D=4)})^{\top}.$$

3.2 $H_{\text{gh}}(s_{\text{gh}})$ for $N = 2$

We define $\mathfrak{so}(t, 5-t)$ -covariant ghost polynomials θ_{ij} and $\theta_{a ij}$ according to

$$\theta_{ij} = c^a c^b \xi_i \Gamma_{ab} C^{-1} \xi_j^{\top}, \quad \theta_{a ij} = c^b \xi_i \Gamma_{ba} C^{-1} \xi_j^{\top} \quad (3.7)$$

with

$$\Gamma_{ab} = \frac{1}{2}(\Gamma_a \Gamma_b - \Gamma_b \Gamma_a). \quad (3.8)$$

In terms of $D = 4$ objects, one has for $i, j \in \{1, 2\}$:

$$\theta_{11} = -4k_5 c^5 \Theta_{21(D=4)} + [\vartheta_2^+ \cdot \vartheta_2^+ - \vartheta_1^- \cdot \vartheta_1^-]_{(D=4)}, \quad (3.9)$$

$$\theta_{22} = 4k_5 c^5 \Theta_{12(D=4)} + [\vartheta_1^+ \cdot \vartheta_1^+ - \vartheta_2^- \cdot \vartheta_2^-]_{(D=4)}, \quad (3.10)$$

$$\theta_{12} = 2k_5 c^5 [\Theta_{11} - \Theta_{22}]_{(D=4)} - [\vartheta_1^- \cdot \vartheta_2^- + \vartheta_1^+ \cdot \vartheta_2^+]_{(D=4)}, \quad (3.11)$$

$$\theta_{511} = 2k_5 \Theta_{21(D=4)}, \quad (3.12)$$

$$\theta_{522} = -2k_5 \Theta_{12(D=4)}, \quad (3.13)$$

$$\theta_{512} = k_5 [\Theta_{22} - \Theta_{11}]_{(D=4)}, \quad (3.14)$$

$a \neq 5$:

$$\theta_{a11} = -2k_5 c^5 [\xi_1^+ \Gamma_a C^{-1} (\xi_2^-)^{\top}]_{(D=4)} + [\vartheta_1^- \Gamma_a C^{-1} (\xi_1^+)^{\top} - \vartheta_2^+ \Gamma_a C^{-1} (\xi_2^-)^{\top}]_{(D=4)}, \quad (3.15)$$

$$\theta_{a22} = 2k_5 c^5 [\xi_2^+ \Gamma_a C^{-1} (\xi_1^-)^{\top}]_{(D=4)} + [\vartheta_2^- \Gamma_a C^{-1} (\xi_2^+)^{\top} - \vartheta_1^+ \Gamma_a C^{-1} (\xi_1^-)^{\top}]_{(D=4)}, \quad (3.16)$$

$$\begin{aligned}
\theta_{a12} &= k_5 c^5 [\xi_1^+ \Gamma_a C^{-1} (\xi_1^-)^{\top} - \xi_2^+ \Gamma_a C^{-1} (\xi_2^-)^{\top}]_{(D=4)} + \frac{1}{2} [\vartheta_1^- \Gamma_a C^{-1} (\xi_2^+)^{\top} \\
&\quad + \vartheta_2^+ \Gamma_a C^{-1} (\xi_1^-)^{\top} + \vartheta_2^- \Gamma_a C^{-1} (\xi_1^+)^{\top} + \vartheta_1^+ \Gamma_a C^{-1} (\xi_2^-)^{\top}]_{(D=4)} \quad (3.17)
\end{aligned}$$

with Θ_{ij} and ϑ_i^{\pm} as in equations (2.102). As θ_{ij} and $\theta_{a ij}$ are symmetric in i, j one has $\theta_{21} = \theta_{12}$ and $\theta_{a 21} = \theta_{a 12}$. Using equations (2.104) for $N = 2$, one easily verifies that all polynomials (3.7) are cocycles in $H_{\text{gh}}(s_{\text{gh}})$ for $N = 2$:

$$N = 2: \quad s_{\text{gh}} \theta_{a ij} = 0, \quad s_{\text{gh}} \theta_{ij} = 0 \quad (i, j \in \{1, 2\}). \quad (3.18)$$

We are now prepared to prove the following result:

Lemma 3.1 (Primitive elements for $N = 2$).

The general solution of the cocycle condition in $H_{\text{gh}}(s_{\text{gh}})$ for $N = 2$ is:

$$s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim P(\xi_1, \xi_2) + \theta_{11}P^{11}(\xi_1, \xi_2) + \theta_{12}P^{12}(\xi_1, \xi_2) + \theta_{22}P^{22}(\xi_1, \xi_2) \\ + \theta_{a11}P^{a11}(\xi_1, \xi_2) + \theta_{a12}P^{a12}(\xi_1, \xi_2) + \theta_{a22}P^{a22}(\xi_1, \xi_2) \quad (3.19)$$

with $\theta_{a ij}$ and θ_{ij} as in equations (3.7) and arbitrary polynomials $P(\xi_1, \xi_2)$, $P^{a ij}(\xi_1, \xi_2)$, $P^{ij}(\xi_1, \xi_2)$ in the components of ξ_1 and ξ_2 .

Proof: We define the subspace $\hat{\Omega}$ of ghost polynomials which do not depend on c^5 :

$$\hat{\Omega} = \left\{ \omega \in \Omega_{\text{gh}} \mid \frac{\partial \omega}{\partial c^5} = 0 \right\}. \quad (3.20)$$

Furthermore we define the subspaces Ω_{gh}^p and $\hat{\Omega}^p$ of Ω_{gh} and $\hat{\Omega}$ containing the ghost polynomials with c -degree p , respectively:

$$\Omega_{\text{gh}}^p = \{ \omega \in \Omega_{\text{gh}} \mid N_c \omega = p \omega \}, \quad \hat{\Omega}^p = \{ \omega \in \hat{\Omega} \mid N_c \omega = p \omega \}, \quad N_c = c^a \frac{\partial}{\partial c^a}. \quad (3.21)$$

We study the cocycle condition $s_{\text{gh}}\omega = 0$ separately for the various c -degrees p . As a ghost polynomial is at most linear in c^5 , each polynomial $\omega^p \in \Omega_{\text{gh}}^p$ can be uniquely written as

$$\omega^p = c^5 \hat{\omega}^{p-1} + \hat{\omega}^p, \quad \hat{\omega}^{p-1} \in \hat{\Omega}^{p-1}, \quad \hat{\omega}^p \in \hat{\Omega}^p. \quad (3.22)$$

This gives:

$$s_{\text{gh}}\omega^p = (s_{\text{gh}}c^5)\hat{\omega}^{p-1} - c^5(s_{\text{gh}}\hat{\omega}^{p-1}) + s_{\text{gh}}\hat{\omega}^p. \quad (3.23)$$

As $s_{\text{gh}}c^5$ is a quadratic polynomial in the supersymmetry ghosts, only the second term on the right hand side of (3.23) contains c^5 . We thus obtain

$$s_{\text{gh}}\omega^p = 0 \Leftrightarrow (s_{\text{gh}}\hat{\omega}^{p-1} = 0 \wedge (s_{\text{gh}}c^5)\hat{\omega}^{p-1} + s_{\text{gh}}\hat{\omega}^p = 0). \quad (3.24)$$

Hence, the part $\hat{\omega}^{p-1}$ of a cocycle ω^p is a cocycle in $\hat{\Omega}$. Furthermore, any contribution $s_{\text{gh}}\hat{\eta}^p$ to $\hat{\omega}^{p-1}$ with $\hat{\eta}^p \in \hat{\Omega}^p$ can be removed from ω^p by adding the coboundary $s_{\text{gh}}(c^5\hat{\eta}^p)$ owing to $\omega^p + s_{\text{gh}}(c^5\hat{\eta}^p) = c^5(\hat{\omega}^{p-1} - s_{\text{gh}}\hat{\eta}^p) + \hat{\omega}^p$ with $\hat{\omega}'^p = \hat{\omega}^p + (s_{\text{gh}}c^5)\hat{\eta}^p \in \hat{\Omega}^p$ redefining the part $\hat{\omega}^p$ of ω^p . Hence, $\hat{\omega}^{p-1}$ is actually determined by the cohomology of s_{gh} in $\hat{\Omega}$ which we denote by $\hat{H}_{\text{gh}}(s_{\text{gh}})$. The latter cohomology can be directly obtained from the results in $D = 4$ since $\hat{\Omega}$ coincides with the $D = 4$ space of ghost polynomials and s_{gh} acts identically in both spaces when the identifications (3.5) are used. In particular, if $\hat{H}_{\text{gh}}(s_{\text{gh}})$ vanishes at c -degree $p - 1$, $s_{\text{gh}}\omega^p = 0$ implies $\omega^p \sim \hat{\omega}^p$. If additionally $\hat{H}_{\text{gh}}(s_{\text{gh}})$ also vanishes at c -degree p , we obtain $\omega^p \sim 0$. Hence, $H_{\text{gh}}(s_{\text{gh}})$ vanishes at c -degree p if $\hat{H}_{\text{gh}}(s_{\text{gh}})$ vanishes at c -degrees $p - 1$ and p . Lemma 2.10 implies that $\hat{H}_{\text{gh}}(s_{\text{gh}})$ vanishes for $N = 2$ at all c -degrees $p > 1$. We conclude immediately that $H_{\text{gh}}(s_{\text{gh}})$ vanishes at all c -degrees $p > 2$:

$$p > 2 : \quad s_{\text{gh}}\omega^p = 0 \Leftrightarrow \omega^p \sim 0. \quad (3.25)$$

In the case $p = 2$ lemma 2.10 implies that the part $\hat{\omega}^1$ of ω^2 can be taken as

$$p = 2 : \hat{\omega}^1 = \Theta_{12(D=4)}H(\xi_1, \xi_2) + \Theta_{21(D=4)}G(\xi_1, \xi_2) + [\Theta_{11} - \Theta_{22}]_{(D=4)}B(\xi_1, \xi_2) \quad (3.26)$$

with polynomials H, G, B in the components of the supersymmetry ghosts. Equations (3.9) to (3.11) show that $c^5\hat{\omega}^1$ can be completed to the cocycle $\theta_{11}P^{11}(\xi_1, \xi_2) + \theta_{12}P^{12}(\xi_1, \xi_2) + \theta_{22}P^{22}(\xi_1, \xi_2)$ wherein

$$P^{11} = -(4k_5)^{-1}G, \quad P^{12} = (2k_5)^{-1}B, \quad P^{22}(\xi_1, \xi_2) = (4k_5)^{-1}H.$$

Leaving out the arguments of the P^{ij} , this yields for the cocycles $\omega^2 \in \Omega_{\text{gh}}^2$ the intermediate result $\omega^2 = \theta_{11}P^{11} + \theta_{12}P^{12} + \theta_{22}P^{22} + \hat{\omega}'^2$ where $\hat{\omega}'^2 \in \hat{\Omega}^2$ is the difference of the part $\hat{\omega}^2$ in $\omega^2 = c^5\hat{\omega}^1 + \hat{\omega}^2$ and those terms in $\theta_{11}P^{11} + \theta_{12}P^{12} + \theta_{22}P^{22}$ which do not depend on c^5 . As $\theta_{11}P^{11} + \theta_{12}P^{12} + \theta_{22}P^{22}$ is a cocycle by itself, the cocycle condition $s_{\text{gh}}\omega^2 = 0$ imposes $s_{\text{gh}}\hat{\omega}'^2 = 0$. The latter implies that $\hat{\omega}'^2$ is trivial in $\hat{H}_{\text{gh}}(s_{\text{gh}})$ since $\hat{H}_{\text{gh}}(s_{\text{gh}})$ vanishes at c -degree $p = 2$ according to lemma 2.10. We conclude in the case $p = 2$:

$$s_{\text{gh}}\omega^2 = 0 \Leftrightarrow \omega^2 \sim \theta_{11}P^{11}(\xi_1, \xi_2) + \theta_{12}P^{12}(\xi_1, \xi_2) + \theta_{22}P^{22}(\xi_1, \xi_2). \quad (3.27)$$

The case $p = 1$ is somewhat more involved and we shall discuss it without giving all steps in explicit details. In the case $p = 1$ the part $\hat{\omega}^0$ of a cocycle $\omega^1 = c^5\hat{\omega}^0 + \hat{\omega}^1$ is purely a polynomial in the components of the supersymmetry ghosts,

$$\omega^1 = c^5\hat{\omega}^0(\xi_1, \xi_2) + \hat{\omega}^1, \quad \hat{\omega}^1 \in \hat{\Omega}^1. \quad (3.28)$$

Hence, the condition $s_{\text{gh}}\hat{\omega}^0 = 0$ in (3.24) is trivially fulfilled and (3.24) only imposes $(s_{\text{gh}}c^5)\hat{\omega}^0 + s_{\text{gh}}\hat{\omega}^1 = 0$. This yields explicitly in the case $N = 2$:

$$i(k_5)^{-1}[\xi_1^+ \cdot \xi_2^+ + \xi_1^- \cdot \xi_2^-]_{(D=4)}\hat{\omega}^0(\xi_1, \xi_2) = s_{\text{gh}}(-\hat{\omega}^1), \quad \hat{\omega}^1 \in \hat{\Omega}^1. \quad (3.29)$$

Equation (3.29) imposes that the left hand side of this equation is trivial in $\hat{H}_{\text{gh}}(s_{\text{gh}})$. Now, any ghost monomial in $\hat{\omega}^0$ which depends both on at least one of the components $\xi_i^{+\alpha}$ and on at least one of the components $\xi_i^{-\alpha}$ yields on the left hand side of equation (3.29) only terms which are trivial in $\hat{H}_{\text{gh}}(s_{\text{gh}})$ as in the case $N = 2$ one has $2i[(\xi_1^+ \cdot \xi_2^+)\xi_2^{-\alpha}]_{(D=4)} = s_{\text{gh}}\vartheta_{1(D=4)}^{-\alpha}$ etc., see equations (2.104). In contrast, ghost monomials in $\hat{\omega}^0$ which do not depend on any of the components $\xi_i^{+\alpha}$ or on any of the components $\xi_i^{-\alpha}$ would provide contributions to the left hand side of equation (3.29) which do not depend on any of the components $\xi_i^{+\alpha}$ or on any of the components $\xi_i^{-\alpha}$. The latter contributions to the left hand side of equation (3.29) would not be trivial in $\hat{H}_{\text{gh}}(s_{\text{gh}})$ because $s_{\text{gh}}c^1, \dots, s_{\text{gh}}c^4$ only contain monomials which depend both on one of the components $\xi_i^{+\alpha}$ and on one of the components $\xi_i^{-\alpha}$. This implies that all ghost monomials in $\hat{\omega}^0$ depend both on at least one of the components $\xi_i^{+\alpha}$ and on at least one of the components $\xi_i^{-\alpha}$. Hence, with no loss of generality, $\hat{\omega}^0$ can be taken as

$$p = 1 : \hat{\omega}^0 = \sum_{i=1}^2 \sum_{j=1}^2 \xi_i^{+\alpha} \xi_j^{-\beta} P_{\underline{\alpha}\underline{\beta}}^{ij}(\xi_1, \xi_2) \quad (3.30)$$

with polynomials $P_{\underline{\alpha}\underline{\beta}}^{ij}(\xi_1, \xi_2)$ in the components of the supersymmetry ghosts. The sixteen ghost monomials $\xi_i^{+\alpha}\xi_j^{-\beta}$ in (3.30) provide twelve independent cocycles in $\hat{H}_{\text{gh}}(s_{\text{gh}})$ because $s_{\text{gh}}c^1, \dots, s_{\text{gh}}c^4$ give four exact linear combinations of these monomials. These twelve cocycles can be taken as the twelve ghost polynomials multiplied by c^5 in equations (3.15) to (3.17). This gives, analogously to the case $p = 2$, $\omega^1 \sim \sum_{a=1}^4 [\theta_{a11}P^{a11}(\xi_1, \xi_2) + \theta_{a12}P^{a12}(\xi_1, \xi_2) + \theta_{a22}P^{a22}(\xi_1, \xi_2)] + \hat{\omega}'^1$ with $\hat{\omega}'^1 \in \hat{\Omega}^1$ and $s_{\text{gh}}\hat{\omega}'^1 = 0$. According to lemma 2.10, $s_{\text{gh}}\hat{\omega}'^1 = 0$ implies $\hat{\omega}'^1 \sim \Theta_{12(D=4)}H'(\xi_1, \xi_2) + \Theta_{21(D=4)}G'(\xi_1, \xi_2) + [\Theta_{11} - \Theta_{22}]_{(D=4)}B'(\xi_1, \xi_2)$. By equations (3.12) to (3.14) this yields $\hat{\omega}'^1 \sim \theta_{511}P^{511}(\xi_1, \xi_2) + \theta_{512}P^{512}(\xi_1, \xi_2) + \theta_{522}P^{522}(\xi_1, \xi_2)$ with $P^{511} = (2k_5)^{-1}G'$, $P^{512} = -(k_5)^{-1}B'$, $P^{522} = -(2k_5)^{-1}H'$. We conclude:

$$s_{\text{gh}}\omega^1 = 0 \Leftrightarrow \omega^1 \sim \theta_{a11}P^{a11}(\xi_1, \xi_2) + \theta_{a12}P^{a12}(\xi_1, \xi_2) + \theta_{a22}P^{a22}(\xi_1, \xi_2). \quad (3.31)$$

The case $p = 0$ is trivial as any element ω^0 of Ω_{gh}^0 is a polynomial $P(\xi_1, \xi_2)$ in the components of the supersymmetry ghosts. Together with (3.25), (3.27) and (3.31) this proves the lemma. \blacksquare

Comment: The decomposition (3.2) of $D = 5$ supersymmetry ghosts is not $\mathfrak{so}(t, 5-t)$ -covariant and, therefore, it was only used in intermediate steps within the derivation of the results in $D = 5$ from results in $D = 4$. Nevertheless, one may use this decomposition in any particular spinor representation to remove redundant cocycles $P + \theta_{ij}P^{ij} + \theta_{a ij}P^{a ij}$ in (3.19) by restraining the ghost polynomials P , P^{ij} , $P^{a ij}$ analogously to lemma 2.10. For instance, one may always assume that P^{11} and the P^{a11} do not depend on the components of ξ_2^- , that P^{22} and the P^{a22} do not depend on the components of ξ_1^+ , that P^{12} and the P^{a12} do not depend on the components of ξ_1^+ or ξ_2^- , and that P does not contain terms depending both on components of ξ_1^+ and on components of ξ_2^- . By refining the proof of lemma 3.1 accordingly, this can be deduced directly from lemma 2.10 owing to the identifications (3.5) and analogously for the $P^{a ij}$ with $a \neq 5$. Hence, in any particular spinor representation one may specify the result (3.19) according to:

$$\begin{aligned} s_{\text{gh}}\omega = 0 \Leftrightarrow \omega \sim & P_+(\xi_2^+, \xi_1) + P_-(\xi_1^-, \xi_2) \\ & + \theta_{11}P^{11}(\xi_2^+, \xi_1) + \theta_{12}P^{12}(\xi_1^-, \xi_2^+) + \theta_{22}P^{22}(\xi_1^-, \xi_2) \\ & + \theta_{a11}P^{a11}(\xi_2^+, \xi_1) + \theta_{a12}P^{a12}(\xi_1^-, \xi_2^+) + \theta_{a22}P^{a22}(\xi_1^-, \xi_2). \end{aligned} \quad (3.32)$$

We leave it to the interested reader to further specify this result or to characterize the remaining coboundaries along the lines of part (ii) of lemma 2.10.

3.3 $H_{\text{gh}}(s_{\text{gh}})$ for $N > 2$

Lemma 3.2 (Absence of primitive elements with c -degrees $p > 1$ for $N > 2$).
 $H_{\text{gh}}(s_{\text{gh}})$ vanishes for $N > 2$ at all c -degrees $p > 1$:

$$N_c \omega^p = p \omega^p, \quad p > 1 : \quad s_{\text{gh}}\omega^p = 0 \Leftrightarrow \omega^p \sim 0. \quad (3.33)$$

Proof: As in the proof of lemma 3.1 we study the cocycle condition $s_{\text{gh}}\omega^p = 0$ by decomposing it according to equations (3.24) and by analysing these equations using the results in $D = 4$. In the cases $N > 2$ lemma 2.11 implies that $\hat{H}_{\text{gh}}(s_{\text{gh}})$ vanishes at all c -degrees $p > 0$. This implies by arguments which led for $N = 2$ to the result (3.25) that $H_{\text{gh}}(s_{\text{gh}})$ vanishes for $N > 2$ at all c -degrees $p > 1$. ■

Conjecture: The author strongly conjectures that $H_{\text{gh}}(s_{\text{gh}})$ vanishes for $D = 5$, $N > 2$ also at c -degree $p = 1$. This would imply that lemma 3.2 holds for all c -degrees $p > 0$ in place of $p > 1$.

4 Conclusion

We have computed the primitive elements of the supersymmetry algebra cohomology for supersymmetry algebras (1.1) in $D = 4$ and $D = 5$ dimensions, for all signatures $(t, D - t)$, all numbers N of sets of Majorana or symplectic Majorana supersymmetries and all spinor representations equivalent to (1.5), except for the particular case of c -degree $p = 1$ in $D = 5$ for $N > 2$ (concerning this case, see the conjecture at the end of section 3.3). The results are given in manifestly covariant form in section 2.2 for $D = 4$ (lemmas 2.9, 2.10 and 2.11) and in sections 3.2 and 3.3 for $D = 5$ (lemmas 3.1 and 3.2). We remark that the seemingly preferred role of the supersymmetry ghosts ξ_1 in lemmas 2.10 and 2.11 originates from our method to base the computations in $D = 4$ for $N > 1$ on the results for $N = 1$, and just provides one particular choice of representatives of the cohomology.

As we have explained in some detail in section 7 of [1], the results of the present work can be used, inter alia, in the context of algebraic renormalization [6], in particular within the classification of counterterms and anomalies, and of consistent deformations [7] of supersymmetric (quantum) field theories in four and five dimensions.

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